A COMPLETE MANUAL

ON TEACHING

ARITHMETIC, ALGEBRA, AND GEOMETRY

INCLUDING

A BRIEF HISTORY OF THESE BRANCHES.

BY

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PREFACE.

The following treatise in its scope and methods is unlike any other work yet offered to the public in this country.

Two distinct lines of thought are developed in the treatment of each branch—the historical phase on the one hand, and the scientific method of presentation on the other. It is believed that the historical features herein presented will be gladly welcomed by all teachers and students of Arithmetic, Algebra, and Geometry, interested in the origin, growth, and present boundaries of these sciences. In the preparation of the historical sketches the writings of Professor De Morgan, President Edward Brooks, Professors Eugène Rouché, and Ch. De Comberousse, and the English and American Encyclopædies were consulted as the most reliable, accessible sources of information. There has not yet appeared a historian of the mathematical sciences; and what has been written is scattered through many volumes, very difficult of access even to the specialist, and entirely beyond the reach of the average reader.

The methods of teaching each subject presented will commend themselves to the discriminating judgment of intelligent teachers everywhere. Each topic is discussed in detail, and the teacher is advised along what lines to work, and how the work may be successfully accomplished. No points are left in obscurity. Important matters are duly emphasized. The language employed is simple, plain, direct, and positive. Non-essentials are kept in the background. The teacher is expected to think, to feel, to act,

and then to inspire his pupils with the same thoughts, feelings, and actions.

The *solutions* of the *difficult Arithmetical* and *Algebraic problems* will be very helpful to a large class of teachers and students who are desirous of mastering some of the intricacies of these exact sciences.

The volume is submitted with the hope that it may stimulate others more competent to carry forward a work of which this is only the beginning.

J. M. Greenwood.

Kansas City, Mo., July, 1890.

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Historical Sketch.

The origin of Arithmetic is unknown, but the term is derived from the Greek word *Arithmos,* number. Different nations have been accredited with the invention of this science. For instance, Josephus says: "For whereas the Egyptians were formerly addicted to different customs, and despised one another's sacred and accustomed rites, and were very angry with one another on that account, Abram conferred with each of them, and confuting the reasons they made use of, every one for his own practices, he demonstrated that such reasonings were vain and void of truth; whereupon he was admired by them in those conferences as a very wise man and one of great sagacity when he discoursed on any subject he undertook; and this not only in understanding it, but in persuading other men also to assent to him. He communicated to them Arithmetic, and delivered to them the science of Astronomy; for, before Abram came into Egypt, they were unacquainted with those parts of learning; and that science came from the Chaldeans into Egypt, and from thence to the Greeks also."

All that can be inferred from the foregoing extract is that Abram had some knowledge of this science, which either originated with the Hebrews or was derived by them from surrounding nations.

For a time the science was supposed to have originated with the Egyptians; others again ascribed it to the Chaldeans; while others gave the credit of it to the Phœnicians because of their nautical skill. Gradually each of these claims has faded away, and modern investigation points unmistakably to its origin as having been in India. It must not be inferred, however, that other nations did not possess some methods of computation or of counting. Counting is certainly coeval with the race. Even "barter," which is the most primitive method of exchange, could not be carried on between individuals without some system of estimating values by "*how many and how much.*" While probably each of the great nations of antiquity had some method of computation, yet the origin and progress of the science, which has now reached a high degree of perfection, belong to India.

The science in its modern form became possible only through the present system of notation. Consequently the most important event in the history of this science was the invention of the denary system. Among the ancient nations which possessed the art of writing, it was natural to represent numbers by letters.

This we see from the Roman method of notation, which is a step in advance of the methods employed by the Greeks and the Hebrews. Both these nations used the first letters of their alphabets to represent numbers from 1 to 10, with the exception that the Greeks inserted a new character to represent "6" so as to conform to the Hebrew notation, since there is no letter in the Greek language corresponding to the sixth letter in the Hebrew.

These two systems correspond "closely, character for

character, up to 80;" yet the Greeks had another notation for inscriptions which resembles the Roman system quite closely. The Roman notation with which we are familiar employs fewer characters than the Greek and admits of more and simpler combinations. Yet it is a clumsy system to work with, notwithstanding its advantages over the notations used by the Hebrews and Greeks.

The Arabic characters have been traced back to the Hindus, who, in turn, claim a divine origin for the invention. As early as the fifth century of the Christian era the nine digits and zero, very nearly in the same form as we now have them, were known to the people of India, and not then as a recent invention, but as holding a permanent position in their literature. Dr. Edward Brooks, in speaking upon this subject, says: "Among the sacred writings of the Hindus there is preserved a treatise upon Arithmetic and Mensuration, written in the Sanscrit language, called Liliwati. This was regarded as of such inestimable value as to be ascribed by them to the immediate inspiration of Heaven. After an introductory preamble and colloquy of the gods, it begins with the expression of numbers by the nine digits and the cipher, or small *o*. The characters are similar to those in present use, and the method of notation is the same. It contains the common rules of Arithmetic and the extraction of the square root as far as two places. The examples are generally very easy, scarcely forming any part of the text, and are written in the margin with red ink. This work is very old, and proves that the Hindus have possessed this system for many centuries. Their knowledge of the science, however, is quite limited. They have no idea of the decimal scale descending, and their management of fractions is tedious and embarrassed."

The general belief, till the discoveries made in Indian Literature, was that the present system of Arithmetic origi-

nated with the Arabs; but the honor of introducing it into Europe belongs to them. It appears that the Arabs obtained a knowledge of Arithmetic either directly from India during the seventh or eighth century, or from the Persians who had received it from the people of India.

As early as the ninth century the Indian system of notation was known to the Arabs. During the succeeding century it was in common use, at about which date it is supposed to have been introduced into Spain by the conquerors and thence spread gradually to the other European countries. However, the sources of information are so conflicting that it cannot be definitely determined at what date it was actually made known in Europe. Some authorities are inclined to place it in the later part of the eighth century or early in the ninth, while others think it at least two centuries later.

Professor Benjamin Greenleaf says: "It is evident that our numeral characters and our method of computing by them were in use among the Arabians about the beginning of the eighth century, when they invaded Spain, and it is probable that a knowledge of them was communicated to the inhabitants of Spain, and gradually to those of the other European countries."

In the Encyclopædia Britannica, Ninth Edition, is this statement: "The method was known to the Arabians in the ninth century, and in the course of the tenth it seems to have come into general use among them, especially in their astronomical tables and their writings. It was probably in the following century that the Arabs introduced the notation into Spain; but in regard to this we have no explicit information, and different accounts are given of the earliest instances of the use of the system in Europe. On the one hand, it is alleged that the figures first occur in a translation of Ptolemy, of the date 1136, while others maintain that they were introduced (about 1252) by means of the

celebrated astronomical tables published by and named from Alphonso the Wise. That their use was known in Italy at the commencement of the thirteenth century appears to be satisfactorily established, for there is no good reason to doubt the genuineness of the MS. writings of Leonardo of Pisa, copies of which have been found bearing dates of 1202 and 1220. Numerous other instances are given of the early use of the nine figures and the cipher, especially by astronomers, and in calendars." (Vol. II., p. 461.)

While this account of the introduction of figures into Europe appears quite reasonable notwithstanding the discrepancies in regard to dates, yet recent investigations tend to show a somewhat different account of the matter. The counter-claim is that the symbols or characters had been brought into Europe before the Arabs invaded Spain.

During the ninth century, it is claimed, the Arabs learned the Indian Arithmetic from a work that is still extant, and that this treatise is founded upon a former collection of works brought from India to Bagdad about the year 773. The Arab treatise was translated into Latin during the Middle Ages, and became known partially to some European scholars. Also, upon careful comparison, it is thought by some that the figures used by the Europeans during the Middle Ages agree very closely with those used by the Arabs in Spain and Northern Africa, while they differ materially from those used by the Arabs in the East. Upon this difference it has been conjectured that the system the Arabs took to Spain was not the one, so far as the form is concerned, as that which they received from India. Again, it is inferred that the Neo-Pythagoreans, who taught the Greeks and Romans the art of "ciphering," had learned directly from the people of India, and that Boethius and his successor used these figures in their mathematical handbooks, and thus they found way by degrees into European schools.

This introduces the characters into Europe from Alexandria. From cumulative evidence it seems quite probable that the Indian Arithmetic was introduced into Europe through two sources: one through Egypt, probably during the fifth or sixth century, and the other passing through Bagdad and from thence into Europe by the way of Spain, somewhat later. That the characters bore a general resemblance, though differing in some respects, to my mind is one of the strongest presumptions in support of a common origin. Exact uniformity was impossible before the art of printing was known. That once discovered, the shapes of the characters are fixed.

At present the earliest writers on Arithmetic, so far as is known, were Greeks. Their writings abound chiefly in matters of speculation.

Pythagoras nearly 2500 years ago attached great importance to numbers. To-day we follow his classification of numbers into Prime and Composite, Perfect and Imperfect, Redundant and Defective, Solid, Triangular, Square, Cubical, and Pyramidal.

Odd numbers he imagined masculine, and even numbers as feminine.

Euclid treats of numbers in the seventh, eighth, ninth, and tenth books of his Elements of Geometry. He discusses proportion, prime and composite numbers; but from the fact that these books are omitted from the Elements, except in Dr. Barrow's edition, the contribution to the science of Arithmetic is of little value.

About a century after, Eratosthenes invented a way of separating prime numbers from composite numbers. He inscribed the series of odd numbers on parchment. Then he cut out such numbers as he found to be composite. The parchment thus cut was called a *sieve*. The method is known in mathematical literature as "Eratosthenes' Sieve."

Diophantus of Alexandria wrote somewhat extensively

upon the properties of numbers. He composed thirteen books or chapters upon this subject; but seven of them were lost or destroyed, so that the contents of six only are really known. Diophantus is, however, much better known as a writer on Algebra than on Arithmetic.

Following Diophantus a century or more later was Boethius, somewhat distinguished as an author, whose production was the classic work in Europe during the Middle Ages, and was regarded as a model for writers to imitate as late as the fifteenth century. This book is simply a curiosity, as compared with our modern treatises upon the subject. Boethius gave no rules for computing by numbers, but confined himself to the discussion of the properties of numbers.

A work in manuscript discovered in the library in Cairo, written by Avicenna, an Arab physican who lived at Bokhara about the year 1100 A.D., is believed to be the first work that employed the Indian characters and the decimal system.

This treatise contains the four fundamental rules, besides many curious and interesting properties of numbers. It is frequently referred to as the oldest text-book, excepting, of course, the Indian treatises.

We now enter upon the printing period of books; and notwithstanding the discovery and propagation of this art, it is difficult to decide who was the author of the first printed treatise on Arithmetic.

Dr. Peacock and others contend that Lucas di Borgo, an Italian monk, is the author of the first printed Arithmetic, called *Summa di Arithmetica,* published in 1494. This is also said to be the first book that used the Arabic characters, but Professor De Morgan is of the opinion that this treatise was preceded by the works of Calandri and Peter Borgo. He admits that the treatise by Lucas di Borgo was the first on Algebra, but that the treatise of Philip Calandri on

Arithmetic was published in 1491, three years before the Arithmetic of Lucas di Borgo.

After the appearance of the first few printed Arithmetics, others appeared in rapid succession, compared to the long intervals between writers of earlier times.

In 1501 John Huswit wrote a small Arithmetic in the German language. It was published at Cologne. He proved the fundamental rules by *casting out the nines.*

Thirteen years later John Kobel published a book at Augsburg. The Arabic figures are inserted, but not used, by this author. He employed counters and the *Roman letters* instead of figures.

The next Arithmetic was published in Paris, in 1515, by Gaspar Lax. The author added nothing new in this volume of about two hundred and fifty pages.

From this time onward there are evidences of original work. The science was added to by one and then by another.

For instance, in 1522, Bishop Toustall, in his "Art of Computation," professes to have read all the books which had been published, and he says there was hardly a nation that did not have such books.

The next writer of note was John Schoner, who edited Regiomontanus's Arithmetic in 1534. Regiomontanus was a celebrated German astronomer whose proper name was Johann Müller. He demonstrated that the number of figures in a cube number could not exceed three times the number of figures in the root.

Jerome Cardan, an Italian physician, mathematician, and author, "celebrated for his science, self-conceit, and absurd vagaries," was born at Pavia in 1501. At the age of 38 he published at Milan his "*Practica Arithmetica*," a work of curious significance. The properties of numbers were treated of according to the manner of his predecessors, and then he dealt somewhat extravagantly upon the

supposed mystic properties of numbers in foretelling future events. These properties he inferred from the numbers that he found in the Scriptures.

This curious discovery is not so much a matter of surprise when it is remembered that "he dealt much in astrology and was a professed adept in the magical arts." His great epoch, says Hallam, is in the science of Algebra.

The first Arithmetic printed in English in 1543 was written by Robert Recorde, an eminent British mathematician, born about the year 1500. This book was entitled "The Ground of Arts: Teaching the Work and Practice of Arithmetic." Recorde's original book has been greatly changed by the editors of different editions. They interlarded the text with their own observations,—so much so as to leave the reader in doubt as to what Recorde originally wrote. A copy of the original edition is in the Greenville Library of the British Museum, and from it Mr. Heppel states that "Recorde calls the unit figure a digit, and the other parts 'articles;' thus, 5437 is made up of the articles 5000, 400, 30, and the digit 7. In the table of contents we find, with more descriptive writing than is necessary to quote here, the main subjects: Declaration of the Profit of Arithmetic; Numeration, where Recorde does not venture beyond ten thousands of millions; Addition; Subtraction; Multiplication; Division; Reduction; Progressions; the Golden Rule; the Backer Rule [inverse proportion]; the Rule of Double Proportion; the use of Fellowship, both with time and without time; and lastly, in Recorde's words: 'Unto all these is added their proof.'"

John Timbs, F.S.A., in speaking of Robert Recorde, uses the following language: "Here should be mentioned the founder of the school of English writers, that is to say, to any useful or sensible purpose,—Robert Recorde, the physician, a man whose memory deserves, on several accounts, a much larger portion of fame than it has met with. He was

the first who wrote on Arithmetic, and the first who wrote on Geometry, in English; the first who introduced Algebra into England; the first who wrote on Astronomy and the doctrine of the sphere in England; and, probably, the first Englishman who adopted the system of Copernicus.

"Recorde was also the inventor of the present method of extracting the square root; the inventor of the sign of equality; and the inventor of the method of extracting the square root of multinomial algebraic quantities. According to Wood, his family was Welsh, and he himself a Fellow of All Souls' College, Oxford, in 1531. He died in 1558 in the King's Bench Prison, where he was confined for debt. Some have said that he was physician to Edward VI. and Mary, to whom his books are mostly dedicated. They are all written in dialogue between master and scholar, in the rude English of the time." ("School Days of Eminent Men," page 118.)

Three inventions are ascribed to Michael Stifel, a Lutheran minister who published his *Arithmetica Integra* at Nuremberg in 1544. He used the signs $+$, $-$, and $\surd$. Stifel acknowledges his obligations to Adam Risca and Christopher Rudolph.

Nicolas Tartaglia ("Stammer"), whose family name is unknown, published a mammoth treatise on Arithmetic and Algebra in the year 1556. It would require a volume to describe it, according to Professor De Morgan.

Humphrey Baker, an English mathematician, published in 1562 "The Well-Spring of Science," a very popular work on Arithmetic.

"Of all works on Arithmetic prior to the publication of Crocker's celebrated book on the same subject (1668), this of Baker's approaches nearest to the masterpiece of that celebrated arithmetician. . . . It continued to be constantly reprinted till 1687, the latest edition we have met with." (Rose's Biog. Dict.)

The author, so it appears, tried to bridge the chasm between the pure abstract properties of numbers upon the one hand and the practical affairs of daily life upon the other. To adapt numbers to commerce and ordinary business was a wide departure, and marks an era in the science.

Simon Stevinus, a Flemish engineer and mathematician, born at Bruges about 1550, made some very important discoveries in arithmetic, algebra, mechanics, and navigation. In 1585 he published at Leyden an Arithmetic, in which he devotes some space to *Interest Tables* and to *Decimals.* This is the first notice of Decimal Fractions.

Albert Girard, another Dutch mathematician, edited Stevin's Arithmetic in 1634, and made such changes as he regarded as desirable. He changed the vinculum for the parenthesis.

John Mellis revised Robert Recorde's "Grounde of Arts," London, 1579, 1582, 1590 (8vo); and in 1588 issued "Bookes of Accompts" (8vo). This is reputed to be the oldest English work on Double-entry Book-keeping.

In an old English work entitled "The Pathway of Knowledge," which was published in London in 1596, are the following lines:

> "Thirtie daies hath September,
> Aprill, June, and November,
> Februarie, eight and twenty alone;
> All the rest thirtie and one."

This book was translated from the Dutch by "W. P.," who is supposed to be the author of the quatrain.

Pietro Antonio Cataldi, an Italian mathematician, was born at Bologna in 1548 and died in 1626. He was professor of mathematics in the university of his native city for more than forty years.

At Bologna he founded an academy of mathematics which

is said to have been the most ancient institution of that kind known, but it was suppressed by the senate.

Cataldi was an original investigator in different branches of mathematics.

His discoveries in Arithmetic are methods of extracting the square root of numbers, and the treatment of continued fractions. New ideas seem to have sprung up in his mind in great profusion, and he occupies a distinguished position among the Italian mathematicians of that century, and his works were used in more than a hundred towns and cities of Italy.

The first English book containing tables of Compound Interest was a work by Richard Witt, published in 1613. This book contained tables which the author called "Breviats," which were used to aid in the solution of problems in compound interest, annuities, rents, and so forth.

The author appears to have used a line for the decimal point; that is, the tables were treated as numerators having 100 . . . for denominator, according to the number of figures in the numerator.

Four years after the appearance of Witt's treatise John Napier published his arithmetic. He claims the invention of the decimal point, and he also states that Stevin (Stevinus) first discovered decimal fractions. Professor De Morgan is of the opinion that Napier did not invent the decimal point, but that he borrowed it from some other author.

Robert Flood, an English physician and writer, born at Milgate in 1574, used P and M with strokes drawn through them. The first was the sign for addition; the second, for subtraction. The work containing these signs was published in 1617–19.

A work written ten years later by Albert Girard does not use the decimal point. His method of expressing 23.375 would be thus: $23\underline{/375.}$ This indicates a carefulness in adopting new inventions.

William Oughtred, an eminent divine and a distinguished mathematician, born in 1573, introduced the sign × (St. Andrew's cross) in his *Clavis Mathematica* in 1631. Benjamin Martin says of Oughtred: "His style was very concise, obscure, and dry, and his rules and precepts so involved in symbols and abbreviations as rendered his mathematical writings very difficult to be understood."

From that time to the present the science of Arithmetic has been perfected in many ways. More authors in England and in America have written on this subject since the days of Oughtred than on any other with the exception, perhaps, of English Grammar.

Preliminary Remarks.

The study of arithmetic should give clearness, activity, intensity, and tenacity to the mind on the disciplinary side; the drill or practical side should train to easy, quick, and accurate computation.

Perception, attention, memory, imagination, judgment, and reason, are quickened and strengthened when the learner has grasped most firmly the fundamental principles of arithmetic, and he can apply them with just discrimination to the solution of problems. The science of number requires the child to deal with things, relations, words, and thoughts. By close attention to these he becomes patient, logical, and systematic—habits of great value in the ordinary affairs of life. Self-mastery of principles is the only sure way to a clear understanding of this subject. Truth is many-sided. Some catch a glimpse from this side, others from that side, and so on. It is the living teacher whose presence, inspiration, directing power, can awaken thought and stimulate a class to its best and highest efforts. Without soul-force, energy, and enthusiasm,—a love for truth and an overweening desire to search for it, to find it and retain it,—all education is naught.

While all true education in that higher sense is the generalization of mental power and noble character, the science of arithmetic is peculiarly adapted to developing continuous and related thought,—in placing before the mind certain definite conditions from which must be deduced necessitated conclusions.

The child begins number concretely at first, but even then the memory and imagination run far ahead of "sense-products." To keep the child too long with "the sensuous and the known" is mental death.

Right instruction in arithmetic requires that training which enables the learner to seize quickly the conditions of a question, and to hold them clearly and firmly and to examine them attentively till *he sees the conclusion.*

The advantages arising from a certain mechanical skill in obtaining results, which is frequently referred to as business or commercial arithmetic, should be secured in connection with the principles of the science rationally apprehended. The shop-keeping idea of arithmetic so prevalent among certain classes, while it may satisfy the superficial, is unworthy the name of the science which it belittles. To study is hard work. To concentrate, to direct, to prolong, to change effort,—to think closely, effectively, and successfully, distinguish the *thinking man* from the *mere man.* To solve problems is beneficial, but to solve problems and to think equally as well on other questions is better still.

To become a good arithmetician is the ideal that should be placed before every one who studies this science. The true teacher is the one who awakens and puts energy, enthusiasm, activity, direction, and confidence into another mind and stimulates it to do its best. The very best work one can do always educates. Striving for something higher, nobler, grander, uplifts the soul and purifies the character.

The following are the essential conditions for teaching arithmetic: 1. *A live, well-qualified teacher, who understands child mind and knows how to teach one thing at a time and how to teach that well.* 2. *A child that can be taught how to sit or stand, how to study, how to think, how to reason, and how to tell or write what he knows.* 3. Books, slates, and pencils, blackboards, crayon, and erasers. 4. Apparatus for illustrations.

Arithmetical Teaching will be presented under three subdivisions: 1. Primary. 2. Mental. 3. Practical and Higher. This classification is somewhat arbitrary in that Mental and Practical and Higher Arithmetic should be pursued simultaneously, and that the Mental in some form is connected with the Primary and Advanced.

PRIMARY ARITHMETIC.

Counting is chiefly a matter of memory at first rather than a desire to find out the number of different objects in a group, or a collection, of things. In my own experience I learned to count a hundred first, and afterwards I counted things to a hundred. Observations with children since confirm this view. This may not, however, be true of all children. Doubtless it is of a large majority. The fact that it exists indicates that tendency of the mind to be always reaching out in regions beyond the known. I hold it to be axiomatic that the child is working nearly all the time with things partially known. The mind soon wearies with the known, and by virtue of its inherent energy is grappling with what is not yet its own. Complete comprehension or thoroughness is not an attribute of childhood; but the child easily makes the transition from words to things, and from things to symbols of things. In counting, which is the first step in arithmetic, the word and the object may be developed together, or the child may learn

first to count, or call the words, "one, two, three," etc., without attaching any meaning to the words beyond the name. I am rather inclined to the opinion that the child when counting objects till the transition is fully made, will call each separate object by the name given it when it was counted. A little girl was counting for me, and an object was counted as "seven." If I took the object, which was an apple, and asked her how many, she replied "seven." So of other numbers. The name of the number was the name of the thing. A difficulty exists at this point, I imagine, with all children. Counting, obviously, is the first thing for the child to learn. Till he can count and attaches some idea to each number, progress in arithmetic is impossible. If the child is unable to count ten, I would practice him on counting till he learned it; then, by memory or repetition, to a hundred. He soon "gets the hang of the thing," and it makes no material difference how he gets it, so he gets it. Some children reach a result one way, and some another. So he has a method that may be the best for him, but not for another whose eyes see in a different direction.

Some minds are quick, but not retentive ; others slower, but retentive ; and a third class very slow, but very retentive. The best teacher always gives the dullest pupil a *large chance.* Nearly all depends, however, upon *the manner in which this chance* is offered.

Returning to counting, all teachers do not follow the same line after the child can count. Personally, I want children to be able to count a hundred, and in teaching primary classes, I always found that they could do so with a little practice, if they had not already learned to count at home. Most children are taught to count before they go to school. Teach them if they do not know how. To count is the preliminary capital that the child must start with. Many teachers begin with objects. For some

children this is unnecessary. They perceive relations without the intervention of objects. Of recent years so many objects have been introduced at the wrong places in arithmetical instruction that great injury has been done to the learners, and their future progress retarded. The proper use of objects for purposes of illustration or of enforcing right ideas, cannot be overestimated. When a truth or a fact is thus once enforced and thoroughly apprehended, it is waste of time to continue illustrations longer. Just how long children should be kept playing arithmetic with objects must be answered by the intelligent teacher. There is perhaps more judgment to be used in knowing when to stop a thing of this kind than in beginning it. Enough is a feast, and too much is worse than a famine. As soon as a child perceives that the words "one, two, three," etc., are used in connection with objects it is time to lessen effort in this direction.

Things to be Observed in Teaching Primary Arithmetic.

1. **The child must be taught to count things.**

2. **The child must learn that words used in counting apply to objects as well as to numbers.**

3. Teach the child the **figures and how to read them and how to make them.**

4. **Teach facts by the use of objects till the child's understanding is thoroughly reached.**

5. Let the child learn the difference between the **spoken word**, the **written word**, the **figure** or **figures**, and the **letter** or **letters** that **represent a number.**

6. Teach one principle or fact at a time.

7. **Teach the child to do his best.**

8. **Use small numbers to illustrate a principle.**

9. **Encourage the child to make his own illustrations.**

10. One illustration properly presented and understood, is better for the ordinary child **than several different illustrations.**

11. Give the child a great deal of mental work. The more the better.

12. Teach the child to think.

13. Teach the child to tell concisely and connectedly what he knows.

14. Teach the child to stand or to sit properly, and to breathe naturally.

15. Let the instruction be of such a character as to cultivate the intellectual faculties and the will.

Teaching Primary Arithmetic.

1. Count objects and develop the idea of numbers, teacher and pupils to provide the objects to be counted. Sticks, books, marbles, pins, window-panes, seats, pupils, etc., can be used.

2. Children must next be taught to write the figures that represent the objects counted. They should make the figures neatly and correctly on the slate, paper, and blackboard, and read them instantly.

3. Let each class exercise on the board be made a slate exercise.

4. When the pupils have learned to read the numbers from 1 to 10, or from 20 to 100, give them exercises to copy on their slates, at the next recitation. Little progress can be made in arithmetic till the learner is able to read and write numbers correctly and rapidly within certain limits.

5. While the child is learning to read and to write numbers, he should be taught how to work with numbers.

6. In his earlier efforts he should work with concrete exercises, each of which should be followed by the same changed into an abstract problem.

First Year's Work.

Reading and Writing Numbers.

I now assume that the learner can read and write numbers to 10, and that he can count to 100. Since he knows 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, it is an easy matter to teach him that 10 and 1 are 11, and so on to 20. That is, he learns all numbers from 10 to 20; and he can recall each at sight. From 20 go to 30; from 30 to 40; from 40 to 50; and so on to 100.

With the intermediate numbers the same plan is pursued, thus: 21 to 31, 31 to 41, 41 to 51, 51 to 61, 61 to 71, and so on. Sometimes the connection is more closely shown, thus: 2, 12, 22, 32, 42, 52, 62, 72, 82, 92, 102; or, 5, 15, 25, 35, 45, 55, and so on.

To teach quickness in reading numbers, the teacher can point rapidly to figures on the board, having one pupil answer while the others watch. It is better to point to several numbers in succession and have a pupil speak each number instantly. Avoid much concert answering. Skipping about is an excellent way to conduct such exercises. The child learns all small numbers just as he learns words—*by their looks.*

It is really more trouble to teach children to read and write numbers from 1 to 100 than for them to learn to read and write from 100 to 1,000,000. After the child is quite familiar with numbers to 100, give him 200, 300, 400, 500, 600, 700, 800, 900, 1000. Then he may afterwards take up 110, 210, 310, 410, 510, 610, 710, 810, 910; and 120, 220, 320, 420, 520, 620; and so on.

There are many ways of reaching the same results. The differences consist in the time it takes some persons to get started, and then to go on after they start. They waste more time in getting ready than is required to teach the pupils the subject from the beginning. Any method that

is given here is simply suggestive. It is to be understood as *a* method and not as *the* method. A tree can be cut so as to fall in almost any direction; the same is true of chopping into arithmetical subjects. Much depends, however, upon the skill of the chopper.

Addition and Subtraction.

The ideas of putting two or more things or numbers into one group, and of separating a collection of things into two or more parts, are developed about the same time. Some go further and teach not only addition and subtraction, but also multiplication and division, simultaneously.

Illustrative Exercises.

1. One apple and one apple are how many apples? Pupil to answer.

2. One and one are how many? Pupil to answer.

3. $1 + 1 = 2$. Now is the time to show how *plus* is used.

4. A boy had two apples and he lost one; how many apples had he left?

5. Two less one are how many? The pupil should now be taught *minus* and *equal to*. That is, $2 - 1 = 1$.

6. Give all such exercises to pupils orally first, and then written out afterwards.

7. Two books and one more book are how many books?

8. Two and one are how many?

9. Two plus one are how many?

10. Write two plus one—equal to what?

11. One, plus one, plus one, are how many?

12. Write this in arithmetical language.

13. Mary picked three roses, and then gave one to Jane; how many roses were left?

14. Three minus one are how many?

15. Write three minus one—equal to what in figures?

These and similar exercises may be continued by adding and then subtracting till 20 or even 30 is reached, or as far as the teacher may desire; first using 1, then 2, 3, 4, 5, 6, 7, 8, 9, 10. Be sure that from use the pupils learn the nature of +, −, =, so that, whether the sign or its equivalent be used, there is no confusion. Arithmetic is a science of signs and symbols, and its language must be learned before much progress can be made.

While working in these simplest of exercises, the teacher will push his pupils far beyond in reading and writing numbers.

Many Primary Arithmetics furnish copious exercises.

Multiplication and Division.

Multiplication and Division rise naturally out of Addition and Subtraction, and doubtless it is owing to this common origin that Grübe and his followers teach the four fundamental operations simultaneously. Instead, however, of teaching each operation as a separate and distinct process, the safer and, perhaps, better method is a compromise between the two. Too many changes introduce confusion, which may be avoided by taking one step safely and securely before attempting a second. The mind is so constituted that it passes readily from one process to its opposite with little effort, and this fact is significant as an educational principle.

Exercises.

1. John bought two apples at one cent each; what did he pay for both?
2. Two times one are how many?
3. $2 \times 1 =$ what? Let the pupil learn the name and the use of $\times$.
4. $1 \times 2 =$?

5. If an apple cost a cent, how many apples will two cents buy ?

6. One is contained in two how many times?

7. $2 \div 1 =$ what? Then, $2 \div 1 = 2$. Teach the name and use of $\div$.

By gradual steps the teacher will lead his class to other concrete problems, followed by abstract ones, using all the smaller numbers first, both as multipliers and divisors.

Thus far the facts of addition, subtraction, multiplication, and division have been partially developed with enough hints to enable the Primary teacher with the aid of any good elementary Arithmetic to start the pupil in the right direction. The theory of work should now be that of many oral and written exercises, involving all four processes. Of course division necessitates the idea of Fractional numbers, which now require attention.

Fractions.

To teach the child to read fractional numbers, especially the simplest forms, is attended with no more difficulty than teaching him to read and write integers. He has the idea of a half, a third, and a quarter long before he enters school. They are words to him denoting things which have a real existence in form. For him to pass from these forms to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{2}{3}$, $\frac{4}{5}$, etc., is an easy transition. As an evidence of this transition, the child knows quite well what is meant by the words half a biscuit, half an apple, half a pie, half a slate, and so of other fractional expressions. All that he really needs to learn is how to read and write them.

In teaching them, I have pursued the following plan quite successfully, teaching $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, $\frac{6}{2}$, $\frac{7}{2}$, $\frac{8}{2}$, etc., till the class can read and write halves instantly. Then connect halves with multiplication and division; as: $2 \times \frac{1}{2} = ?$

$2 \times 1 = ?$ $2 \times 2\frac{1}{2} = ?$ $2 \times 3 = ?$ $1 \times 2 = ?$ $1 \times 1\frac{1}{2} = ?$
$1 \times 3 = ?$ $1 \times 3\frac{1}{2} = ?$ Half of $5 = ?$ $2\frac{1}{2} \times 2 = ?$

Or use concrete exercises, and then the abstract ones. After halves, which may be carried to any reasonable extent, the idea of thirds is introduced and developed ; and fourths, fifths, and sixths follow naturally. The main point is, however, to carry multiplication and division by fractions along with the four fundamental rules till the pupil or class can add and subtract, multiply and divide, fractions as well as integers.

By way of variety the pupil will add and subtract similar fractions, if he is started first with halves, as easily as he does integers. Any teacher can make this experiment and satisfy himself if he be skeptical. Let me urge again that the teacher mix concrete and abstract examples about equally, and he may use larger numbers as the pupils' faculties expand and are prepared to grasp larger and more complex conditions.

So far the object has been to put the child during his first year in school in the way of using numbers intelligently,—laying, as it were, a foundation for him to build upon in future.

It is no more difficult for the child to tell that the half of 5 is $2\frac{1}{2}$ than it is for him to say that "3" is half of "6"; or that "twice ten is twenty," and that the "half of twenty is ten." Doubling up, cutting "in two," are simple processes. The only caution—take short steps, and plan the child's work systematically.

There should not only be a plan or method of teaching a child to read, write, add, subtract, multiply, and divide numbers for the year, but a plan of how to teach each subject at the proper time and in the right way. Random, unsystematic teaching can never be very successful.

General Directions for the First Year.

1. Be sure that the pupil can count objects correctly.

2. That he knows the figures, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, at sight, and that he can make them correctly.

3. Teach Addition, Subtraction, Multiplication, and Division by using small numbers, integral and fractional. Let the pupils illustrate these processes by using objects first, and abstract numbers afterwards.

4. Teach the pupils how to sit and how to stand, and how to put their work in proper form on blackboard, slate, and paper.

5. Give mental exercises for the purpose of leading pupils to do abstract work.

6. Drill daily in reading and writing numbers.

7. Teach the signs $+$, $-$, $\times$, $\div$, $=$, and how to use them.

8. Give many combination exercises during the latter portion of the school year.

9. Teach the pupils to count United States money; and how to measure with their rulers, and how to measure with gill and pint cups.

10. How to read and write numbers to 1000 for minimum work. Roman notation to L.

Pupils have no use for a text-book.

Older pupils will do all this work in one third of the time.

Second Year's Work.

This year's work should be carried forward without using a text-book. There is no need of supplying a child with an Arithmetic before he can read intelligently what the book contains. A manual or guide-book for the teacher to shape her work by is required in all good schools, and it prevents that dissipation of energy which characterizes erratic work. The second year, or that period which cor-

responds to it with older and more mature children, is the time for fixing deep and permanent number forms in the child's mind. There are *number* forms, just as there are *word* forms and *drawing* forms, which need to be learned and retained. This, too, is the period when the child acquires skill in accurate and rapid calculation, if an opportunity is given. Little children try their powers of speed in running and playing, and mentally they have like desires to gratify. Whether they shall always move demurely and soberly, or be let loose occasionally to caper, is a question that the intelligent teacher will not be long in deciding. Premature age and soberness do not well become a lively, frolicsome child. There is something so unnatural about it that the genuine teacher of childhood cannot be in sympathy with it. We should keep our children young as long as possible. The green places in Arithmetic will give a vigorous glow to the body as well as a warm activity to the mind when properly presented by a master.

Addition and Subtraction.

During the first year many teachers lay special stress upon Addition, and all the energy of the pupil is consequently expended upon this subject. As a matter of choice, delay till the second year, or until that stage in the child's development is reached which corresponds to this period, rather than to begin so early. The chief object now is to perfect the child in the work which he has already commenced. For this purpose the teacher should be supplied with several first-class primary arithmetics from which to select problems. But in addition to these aids other work should be included. For instance, the eye and the memory should be trained in the study of arithmetic, as well as the reasoning faculties. The eye must be trained to see quickly and accurately, the memory to hold facts, objects, and re-

lations tenaciously, while the higher faculties deduce conclusions from the data given and retained. Children can be taught to add numbers with as much accuracy as they read the lessons in their Reading Books, instead of blundering over each column of figures two or three times. Correct teaching should eliminate all uncertainty. This can be done by teaching first all integral forms of the nine digits, and secondly, the combinations of the digits themselves.

These should be learned as *number forms,* and recognized instantly by the pupil. These forms are: $\mathbf{1} = 1$, $\mathbf{2} = 1 + 1$, $\mathbf{3} = 2 + 1 = 1 + 1 + 1$, $\mathbf{4} = 2 + 2 = 3 + 1 = 1 + 2 + 1 = 1 + 1 + 1 + 1$, $\mathbf{5} = 4 + 1 = 3 + 2 = 2 + 2 + 1 = 1 + 1 + 1 + 1 + 1$, $\mathbf{6} = 5 + 1 = 4 + 2 = 3 + 3 = 2 + 2 + 2 = 1 + 2 + 2 + 1$, $\mathbf{7} = 6 + 1 = 3 + 4 = 3 + 3 + 1 = 5 + 2 = 4 + 2 + 1$, $\mathbf{8} = 4 + 4 = 3 + 5 = 6 + 2 = 4 + 3 + 1$, $\mathbf{9} = 6 + 3 = 8 + 1 = 5 + 4 = 3 + 3 + 3$.

As an illustration, "6" is "3 + 3," "4 + 2," "5 + 1," or any other combination which produces it. Instead of seeing "4 + 2" as two separate numbers in Addition, they must be seen as "6," and the same is true of all the other digits. "9," for instance, is "7 + 2," "8 + 1," "5 + 4," "6 + 3," or any other combination whose sum is "9." A little judicious practice in this direction will enable a class of small children *to see a number in a group of numbers instantly.* Quick sight and sharp practice are necessary to accomplish this object.

Such questions as the following will enable the teacher to get a clear idea of what is intended in the above.

1. What two numbers make 5? What three numbers make 5? What four numbers make 5? Write all the numbers that make 5.

2. What three numbers make 9? What five numbers?

Six numbers? Write 9 in all the ways that you can. How many ways?

These exercises can be multiplied at pleasure.

Combinations of the Digits.

As the above table includes all necessary combinations to 9, those above 9 will now be considered.

1. $\genfrac{}{}{0pt}{}{5}{5} = 5 + 5 = 6 + 4 = 10.$ **2.** $\genfrac{}{}{0pt}{}{6}{5} = 5 + 6 = 6 + 5 = 11.$ **3.** $\genfrac{}{}{0pt}{}{7}{3} = 7 + 3 = 3 + 7 = 10.$ **4.** $\genfrac{}{}{0pt}{}{4}{7} = 7 + 4 = 4 + 7 = 11.$ **5.** $\genfrac{}{}{0pt}{}{5}{7} = 7 + 5 = 5 + 7 = 12.$ **6.** $\genfrac{}{}{0pt}{}{6}{7} = 7 + 6 = 6 + 7 = 13.$ **7.** $\genfrac{}{}{0pt}{}{7}{7} = 7 + 7 = 14.$ **8.** $\genfrac{}{}{0pt}{}{2}{8} = 8 + 2 = 2 + 8 = 10.$ **9.** $\genfrac{}{}{0pt}{}{3}{8} = 8 + 3 = 3 + 8 = 11.$ **10.** $\genfrac{}{}{0pt}{}{4}{8} = 8 + 4 = 4 + 8 = 12.$ **11.** $\genfrac{}{}{0pt}{}{5}{8} = 8 + 5 = 5 + 8 = 13.$ **12.** $\genfrac{}{}{0pt}{}{6}{8} = 8 + 6 = 6 + 8 = 14.$ **13.** $\genfrac{}{}{0pt}{}{7}{8} = 8 + 7 = 7 + 8 = 15.$ **14.** $\genfrac{}{}{0pt}{}{8}{8} = 8 + 8 = 16.$ **15.** $\genfrac{}{}{0pt}{}{1}{9} = 9 + 1 = 1 + 9 = 10.$ **16.** $\genfrac{}{}{0pt}{}{2}{9} = 9 + 2 = 2 + 9 = 11.$ **17.** $\genfrac{}{}{0pt}{}{3}{9} = 9 + 3 = 3 + 9 = 12.$ **18.** $\genfrac{}{}{0pt}{}{4}{9} = 9 + 4 = 4 + 9 = 13.$ **19.** $\genfrac{}{}{0pt}{}{5}{9} = 9 + 5 = 5 + 9 = 14.$ **20.** $\genfrac{}{}{0pt}{}{6}{9} = 9 + 6 = 6 + 9 = 15.$ **21.** $\genfrac{}{}{0pt}{}{7}{9} = 9 + 7 = 7 + 9 = 16.$ **22.** $\genfrac{}{}{0pt}{}{8}{9} = 9 + 8 = 8 + 9 = 17.$ **23.** $\genfrac{}{}{0pt}{}{9}{9} = 9 + 9 = 18.$

Drill the class on these combinations till they can tell the sum of any two numbers instantly. The above table goes more into detail than is necessary, perhaps; but "18," for instance, must be seen in all the forms under which "two 9's" can possibly appear. To teach these forms the

teacher can put on the board all combinations, and then pointing to each in rapid succession, receive prompt, accurate replies. When the child looks at $\genfrac{}{}{0pt}{}{7}{6}$, or $\genfrac{}{}{0pt}{}{9}{4}$, or any other combination that is "13," he should see "13" instantly as one number without going through the slow steps of addition by seeing and thinking over each number separately. Ten minutes' drill each day for a month or two will give the average pupil complete mastery of these simple combinations. The time will be profitably spent.

Another excellent exercise in Addition consists in calling a class of pupils in front of the blackboard, and then having all stand with their backs to the board while the teacher writes a column of figures on the board to be added. At a given signal all the pupils face the board for a few seconds, and at a second signal each writes the answer, or retains it mentally. No pupil should have a chance to add the column the second time. Begin with a few figures in a column at first, and thus the pupils gain confidence in themselves. The most rapid additions I have ever seen, with the exception of one or two "Lightning Calculators," were by little children in the *first* and *second* year's work in city schools. They were "*double-geared mental lightning let loose*," and—the best of it all—each one added correctly. Dreamy, slow drilling never accomplishes anything. Sharp, quick, active, intelligent work tells.

As a variation in "seeing, retaining, and adding numbers," the following is eminently practical: Call the class in front of the board, standing with faces from the board. The teacher now writes a single column of figures on the board, and at a given signal all the class turn with faces to the board for an instant, and then at a signal turn from it, and at another signal write or speak the *sum*. Children with a little practice daily will take in a column of eight or ten figures in one or two seconds, and will give the correct

result nine times out of ten. Children that have been well trained in "seeing and adding" will add a column that amounts to 80 or 100 in four seconds. This speed is frequently attained by "straight addition," that is, taking each figure separately without grouping.

For several years I have made a special study of *how children add.* The artifices they adopt from necessity form an amusing bit of experience in school life.

A description of one exercise will be given which occurred recently. There were more than fifty children in the room —second grade. The teacher had drilled the children quite well; that is, they added a column of several figures rather rapidly. She believed they added without counting, because she had told them not to count, make marks, or do anything of that kind. The children looked away while she wrote a column of figures on the board, and when the word was given they turned their faces to the board and began to add. The position I occupied gave me command of each face, and the movements of lips, eyes, and fingers betrayed the workings of each mind. More than half the pupils used "some dodge;" but the teacher had discovered nothing, and when I asked her if she was satisfied that her children added the figures directly, she replied, "Certainly!" She was requested to stand where she could see every face while I gave the class a similar exercise. This she did eagerly, and noted the results.

A few of the artifices are given. A boy had written on a card the digits from 1 to 9.

If he wanted to add "7," say, he counted on his "keyboard" to 7, always beginning with "1" and actually counting up to the required number. It was absolutely astonishing how rapidly he counted. He made good time.

A second boy added by "twos"—his father had taught him, and he broke all digits above two into *twos;* and a third child added by *threes.*

A little German girl, if required to add $8 + 9 + 7$, would take 1 from 8, and 2 from 9, and then say "21, 24." That is, she added by "sevens"—the only case of that kind I ever found.

Counting fingers, buttons, marks, etc., were also employed by some. Without exaggeration, I am inclined to the opinion that many children use some artifices in Addition. That they do resort to aids which they either devise or pick up is an evidence of erroneous instruction in the beginning. To start right and to keep right is the only way not to lose time in school work as well as in other pursuits.

Since the pupil should be able to tell the sum of any two digits instantly, the teacher should not relax vigilance in number forms, but continue the work systematically and vigorously. Each step should be so well planned in Addition as to fix the child's attention by its beauty and its novelty. Without saying so, it teaches a law by its exactness. Thus: 1.

$$\begin{array}{r} 1 \\ 1 \\ \hline 2 \end{array}, \quad \begin{array}{r} 11 \\ 1 \\ \hline 12 \end{array}, \quad \begin{array}{r} 21 \\ 1 \\ \hline 22 \end{array}, \quad \begin{array}{r} 31 \\ 1 \\ \hline 32 \end{array}, \quad \begin{array}{r} 41 \\ 1 \\ \hline 42 \end{array}, \quad \text{etc.}$$

The right-hand digit in the sums is the same.

Again, 2. $$\begin{array}{r} 2 \\ 1 \\ \hline 3 \end{array}, \quad \begin{array}{r} 12 \\ 1 \\ \hline 13 \end{array}, \quad \begin{array}{r} 22 \\ 1 \\ \hline 23 \end{array}, \quad \begin{array}{r} 32 \\ 1 \\ \hline 33 \end{array}, \quad \begin{array}{r} 42 \\ 1 \\ \hline 43 \end{array}, \quad \text{etc.}$$

In a similar manner use 3, 4, 5, 6, 7, 8, 9.

To illustrate: $$\begin{array}{r} 6 \\ 9 \\ \hline 15 \end{array}, \quad \begin{array}{r} 16 \\ 9 \\ \hline 25 \end{array}, \quad \begin{array}{r} 26 \\ 9 \\ \hline 35 \end{array}, \quad \begin{array}{r} 36 \\ 9 \\ \hline 45 \end{array}, \quad \begin{array}{r} 46 \\ 9 \\ \hline 55 \end{array}, \quad \begin{array}{r} 56 \\ 9 \\ \hline 65 \end{array}, \quad \text{etc.}$$

And, $$\begin{array}{r} 8 \\ 9 \\ \hline 17 \end{array}, \quad \begin{array}{r} 18 \\ 9 \\ \hline 27 \end{array}, \quad \begin{array}{r} 28 \\ 9 \\ \hline 37 \end{array}, \quad \begin{array}{r} 38 \\ 9 \\ \hline 47 \end{array}, \quad \begin{array}{r} 98 \\ 9 \\ \hline 107 \end{array}, \quad \text{etc.}$$

The method is this: Add the digit to a series of numbers which increases by ten till the child sees the *law.* It is teaching one thing at a time.

After each row of additions call attention to the unit's figure, thus:

$$\frac{\begin{array}{r}9\\9\end{array}}{18},\quad \frac{\begin{array}{r}19\\9\end{array}}{28},\quad \frac{\begin{array}{r}29\\9\end{array}}{38},\quad \frac{\begin{array}{r}39\\9\end{array}}{48},\quad \frac{\begin{array}{r}49\\9\end{array}}{58},\quad \frac{\begin{array}{r}59\\9\end{array}}{68},\quad \frac{\begin{array}{r}69\\9\end{array}}{78},\quad \text{etc.}$$

The unit "9" is the important point. Fix it deep in the mind. This is designed to impress forms that must be absolutely remembered for all time. The teacher can multiply exercises at pleasure.

Reading and Writing Numbers.

The processes of Reading and Writing Numbers should now be carried forward with energy. Practically there is no limit within reason that can be assigned as a fixed boundary on this point, and beyond which pupils cannot pass. Problems in addition, subtraction, multiplication, and division should be continued. The exercises should be of such a character as to test the pupil's knowledge and skill—just severe enough to make him do his best. The teacher must avoid too simple exercises. They enfeeble the pupil's powers. One problem that makes a child think is worth more than a hundred easy problems. Not-work hurts children. They are made dull, ignorant, lifeless, when kept doing that which amounts to nothing. To wear children out, confine them, and then give them easy, silly things to do.

In addition, subtraction, multiplication, and division, use larger abstract numbers usually, rather than concrete numbers. Abstract numbers should be employed to secure accuracy; concrete examples to develop the thinking facul-

ties. In the explanations of concrete problems, pictorial diagrams and other graphic methods of illustration should be employed by the pupils. All work of whatever character ought to be neatly done.

During this year's work is the proper time to commence teaching the tables of length, weight, and measure. The foot rule with which all pupils are supplied, can be used to teach linear measure. The pupils ought to measure the lengths of familiar objects in the school-room till they can do so with considerable exactness. Two or three recitations will be enough to show them how to measure; or rather, the teacher should see that they do it well. Rulers having the metric system on one side and our common system on the other are to be preferred. But at first the children should use the common measure. Let the children estimate heights and distances by the eye. Avoirdupois weight is best taught with "the scales." Let the children learn to weigh articles at the grocery-store if in no other way. A gill, a quart, and a gallon measure, with a bucket of water, are sufficient to teach "liquid measure." Here, too, the children should do the "dipping and pouring."

From these suggestions the skillful teacher can devise other methods to aid in this work.

The following will illustrate what was suggested in regard to the graphic representation of problems.

Problem. There were two gallons of molasses in a tin can; a pint leaked out, and three quarts were used; how many pints remained in the can?

This problem may be represented pictorially. 1. A tin can drawn to a scale to represent two gallons. 2. A quart measure drawn to a scale to indicate that vessel. 3. A pint cup drawn to a scale. 4. Let the pupil show by drawings how many quarts, or how many pints, two gallons equal. 5. Let him show by lines what part of two gallons one

pint is. 6. What part of two gallons three quarts are. 7. What part of the two gallons is that remaining in the tin can?

Exercises in Fractions.

This work should be continued from that of the preceding year, but it should take a wider scope, and much of the work ought to be oral rather than written.

To illustrate: If a pound of ginger costs 9 cents, what will two pounds cost? What will a half-pound cost? What will a third of a pound cost? What will a fourth of a pound cost? A fifth of a pound? A sixth of a pound? Etc.

Again: If a boy can chop a cord of wood in 10 hours, how many hours will it take him to chop a half-cord? A fourth of a cord? A cord and a third? Five eighths of a chord? Etc.

In addition to solving problems containing fractions, the pupils should have considerable practice in reading and writing fractions, as well as continue addition, subtraction, multiplication, and division. The problems should be selected with reference to teaching methods of work, and to test the pupils' skill in solving them. No problem should be given because it is easy, unless for the purpose of leading to another or others that are more difficult.

Children will work with fractional numbers just as well as with integral numbers if they are put at it in the right way. There is no good reason why fractions should be postponed till the child has been in school three or four years.

Decimals.

Children during this year can be taught to read and to write decimals, as well as to perform the simpler operations

in decimals. The first step is to show that

$$\frac{1}{10} = .1\,, \quad \frac{2}{10} = .2\,, \quad \frac{3}{10} = .3,$$

are more easily written decimally than as common fractions.

The pupil is led to Addition and Subtraction as follows :

$$\frac{1}{10} + \frac{1}{10} = \frac{2}{10} = .2, \text{ etc.}$$

Subtraction : $\frac{3}{10} - \frac{2}{10} = \frac{1}{10} = .3 - .2 = .1.$

Multiplication and Division in a similar manner.

Also, $4\frac{1}{10} = 4.1$; $3\frac{7}{10} = 3.7$; $1\frac{3}{10} = 1.3$; and so on.

Again, $4\frac{7}{10} + 3\frac{7}{10} = 4.7 + 3.7 = 8.4$.

The above are sufficient to suggest how the work can be approached.

The teacher will observe that the pupil is encouraged to solve the problems in a rational manner first, and then to invent a method of illustrating each exercise when it can be done. The teacher must not forget that he is laying a good, broad foundation for the pupil's future mathematical work.

The terms, as such, employed in the fundamental rules ought to be learned by the pupils during the second year. Let nothing be taught that must be unlearned.

Third Year's Work.

This year's work is simply an enlarged continuation of what has been previously outlined. Abstract exercises in small numbers and also in large numbers should constitute a large part of the pupil's work this year. These exercises are designed to give accuracy and rapidity in computation. It is better, at recitation, having ascertained the pupils'

speed of work, to time the class on each problem. The main point is to secure absolute accuracy at the first trial. Much, indeed very nearly all, depends upon quick, sharp, correct training at this stage of the learner's progress. Do not be afraid of giving the children large numbers in addition, subtraction, multiplication, and division to work with. In division, the exercises need to be more carefully graded than in the other rules. The oral work in addition may include such exercises as $23 + 48 = ?$; $21 + 72 + 34 = ?$, to be added at sight. The pupil is to get at the problem in his own way. But the teacher should help him finally in solving it in the easiest manner. A little practice on such problems will be time well spent. The exercises need not be confined to addition. Some time during the year an Elementary Arithmetic is usually introduced for the pupils to study. Since pupils waste two thirds of their time in arithmetic in trying to understand just what is required in the problems, considerable time may be profitably occupied in having pupils at recitation read problems and explain them. When a child knows what is to be done, usually all doubt is removed, and this is really the important step in solving problems, and it is one reason why certain pupils succeed so much better than others in arithmetic. They interpret problems more understandingly. The best way, perhaps, to find out this fact is to assign problems for the pupils to tell what conditions are given, and what required. The proper questions are: What does the problem tell? What is to be found out? How will you go to work to find it out? The power of correctly interpreting what is written or printed is a matter of great importance in teaching. Problems given orally by the teacher usually have this advantage over printed conditions—that of the teacher's voice, which states the conditions more clearly and forcibly, and the emphatic points are more easily perceived. The teacher

vivifies the problem, sets it in a strong light before the children's minds, and they seize all its conditions readily. In expressed problems, such as

$$864 \times 15 - 361 \div 19 + 324 - 15\tfrac{3}{4} \times 4 = ?,$$

the pupil must be taught to interpret the mathematical language correctly. To know whether the pupil understands what he reads, he should be called upon to read it, and to explain it.

If the children are studying one text-book, the teacher should require them to read from other sources similar problems, and to tell what each problem means.

Terms and Signs.

The following terms need to be thoroughly impressed : *Sum, Amount, Minuend, Subtrahend, Difference, Remainder, Multiplicand, Multiplier, Product, Dividend, Divisor, Quotient, Fraction, Numerator, Denominator, Line, Surface, Area, Volume, United States Money, Measures of Length, Liquid Measure, Dry Measure, Avoirdupois Weight, Time by Clocks and Watches.*

Making Change.

The pupils have now reached that stage in Arithmetic when they should be able to *make change,* or to know, when they purchase articles, the amount of money the article or articles cost, and what they are to receive in return. The teacher will show the pupils how to make out an itemized statement of a Bill of Goods, and how to receipt it. This is best illustrated by making out simple purchases at first. A purchase of a few things at a grocery-store, properly itemized, will serve as a model, namely: George Miller bought of Thomas Jones 3 pounds of coffee at 15 cents a pound, 6 pounds of sugar at 10 cents a pound,

$\frac{1}{2}$ bushel of potatoes at 80 cents a bushel. He gave Thomas Jones a two-dollar bill in payment. What did the articles cost, and how much in change should Thomas Jones hand George Miller?

Form.

Kansas City,
September 20, 1889.

George Miller,
Bought of Thomas Jones.

1889.		
Sep. 20.	To 3 pounds of coffee at 15 c. a pound,	.45
" 20.	" 6 pounds of sugar at 10 c. a pound,	.60
" 20.	" $\frac{1}{2}$ bushel of potatoes at 80 c. a bush.,	.40
		$1.45

Other exercises can be multiplied at pleasure from actual business transactions. The children should be encouraged to make problems of their own, and to put them into proper form. Let the work be done neatly.

In the tables and their applications in Compound Numbers the pupil should see and handle the various measures in order to get clearer and better conceptions of the terms used in measurement. To correct their ideas of size, distance, weight, volume, etc., they "should guess" at objects, such as pitchers, buckets, boxes, barrels, etc., and then find out what each holds.

This is really the measuring period in arithmetical studies, and the pupils need to make the most of it. Size, weight, height, distance, and volume of objects can be closely approximated after the pupils have had considerable experience in correcting their own judgment in such matters.

Strengthen weak pupils in reading and writing numbers,

first in integers, and secondly in fractions—both common and decimal. Much practice and little theory should be the motto now. Teach pupils to work problems well *one way*. Make each pupil strong in addition and subtraction, and very strong in multiplication and long division. Let the work half the time be mental, judiciously mixed with written work. Devices, such as are found in several elementary arithmetics for securing rapid computation, should be employed by the teachers. Be sure that the multiplication-table is known with unerring certainty.

What has been marked out as a course of study in Arithmetic for three years applies to young children entering and continuing in graded schools; but a child ten or twelve years old, or older, can do all this work and even more in one year.

In fact, a great deal of the little work can be omitted entirely with older children.

Fourth Year's Work.

It is still necessary to review notation and numeration, and to impress the difference between the *simple* and *local* values of the digits. Some time should be given to reading and writing the Roman notation. The work in *Bills and Accounts*, carried out in extended statements, should be continued from the previous grade. This work will be furthered if the pupils, instead of fictitious persons, are made parties to the transactions.

In this connection the following terms, *Buyer, Seller, Creditor, Debtor, Credit, Debit, Account, Balance, Statement, Payment, Receipt, To,* @, ought to be mastered by the pupil, and then embodied in the exercises. When the idea dawns upon the pupil's mind that his knowledge of arithmetic can be applied to business matters, he is

stimulated to greater effort because of the use he is able to make of what he already knows. But at no stage of the pupil's progress will it do to relax effort in the four fundamental rules. Abstract numbers are to be used chiefly to secure accuracy and expertness. Let the teacher bring out by contrast the differences as expressed by *whole numbers, denominate numbers, fractional numbers,* and *decimal numbers.* By a little reflection the pupil learns that he can add, subtract, multiply, and divide each of these numbers. When working with a number, the pupil must keep in mind the kind and meaning of the number, and the relation it bears to other numbers of a higher or lower denomination. This involves the comparison of one number with others, which cannot be effected unless the idea of similarity or likeness be clearly comprehended.

This is the period in the learner's work when he must be thoroughly drilled in fractions, factoring, cancellation, and reduction. Unless he reads and writes numbers readily, and is accurate and rapid in handling numbers in the four fundamental rules, he will make no headway in fractions. After the pupil is thoroughly familiar with the processes of reduction of fractions, and knows exactly how to make dissimilar fractions similar, so as to add, subtract, and divide them, he should compare simple addition, compound addition, and addition of fractions, and note particularly how these processes agree with one another. In a similar manner, he should trace the comparison through subtraction, multiplication, and division. Let problems be selected to illustrate these several relations. As soon as a pupil finds out that he can hitch new knowledge to what he already has, he has made a rapid stride forward in his generalizations. Following in the line of work is the subject of decimals. Here again let the analogies in the four fundamental operations in simple numbers and the corresponding ones in decimals be discussed. To show the con-

nection between common fractions and decimals, simple exercises, as

$$\frac{1}{10} = .1, \quad \frac{2}{10} = .2, \quad \text{etc., and} \quad \frac{1}{100} = .01, \quad \frac{5}{100} = .05,$$

etc., are to be used.

The fact to be taught is this: Decimals are common fractions with the *dividing line and denominator omitted.* The decimal point stands for the *line* and the *denominator* of the common fraction.

The two methods of multiplying and dividing fractions deserve special attention. Let the pupil make problems and illustrate the methods. Be sure that these operations are mastered. Whenever the teacher calls for a specific thing to be done in definite arithmetical language, the pupil should know exactly what is meant and how to proceed. Precision in language and in thought are of the utmost importance in all mathematical studies. If a good foundation is laid in the fundamental rules, factoring, cancellation, reduction, common and decimal fractions, and an intelligent application of these rules to the measurement of length, area, and volume of forms, the pupil is prepared to take up the simpler cases in percentage.

The necessity, however, of understanding decimals well before beginning percentage is so apparent that special attention is called to this subject. The language of decimal notation and numeration is positive, and must have a permanent place in the learner's mind. All vagueness must disappear and positive, definite, sharp-cut knowledge take its place. On all points the teacher *must know* that the *learner knows* how to write, read, add, subtract, multiply, and divide decimals with absolute accuracy. The placing of the decimal point properly is the important matter in this subject. If the pupil can read and write decimals, he

will experience little difficulty in addition and subtraction. But in multiplication he must watch for the decimal places in multiplicand and multiplier and product; and in division, when the decimal places in the dividend exceed those in the divisor, or when equal or less, and the reverse, and the effect on the quotient under all possible conditions. *In short, the pupil must know exactly where to put the decimal point* in every operation.

Percentage and Interest.

The pupil adds new terms now to his arithmetical vocabulary, and it saves time to master the following at the outset: *Per Cent; Rate Per Cent; Sign of Per Cent* (%)*; Base; Percentage.* If the pupil does not get the meaning from the definitions in his book, the teacher should make the terms plain to the pupil. The key to percentage is this: That 1 per cent of a number is $\frac{1}{100}$ part of that number, $2\% = \frac{2}{100}$, $3\% = \frac{3}{100}$, etc.; or, decimally, $1\% = .01 = \frac{1}{100}$, $2\% = .02 = \frac{2}{100}$, $3\% = .03 = \frac{3}{100}$, etc. Let the pupil express "per cent" in these three ways till he knows each well.

Another point to dwell upon is that of fractional equivalents translated into "per cents," and the reverse.

When 10% is mentioned, the pupil should think along with that $\frac{1}{10}$; 20%, $\frac{1}{5}$; 30%, $\frac{3}{10}$; 5%, $\frac{1}{20}$; and so on. Frequently it is simpler to use the fractional equivalents, because they are smaller.

To illustrate and impress the three cases of percentage, the greatest variety of problems ought to be used, and the pupils required to work them out under each of the three ways of expressing the notation.

Under Interest, the "Six Per Cent Method," owing to its simplicity and universal application, is the one that should be taught first. When it is mastered the pupil

can find the interest at other rates by increasing or decreasing the interest at "6%."

It is better to teach this method well and have the pupil use it exclusively for a long time than to teach partially several different methods. To do a thing well in one way is far more valuable to the pupil than to half-way do it in half a dozen ways.

In connection with Interest the pupils should be instructed how to write *notes* and how to make *credits.* When the pupils are parties to the transactions a reality is given to the work that is very helpful.

If a class is strong and up fully with its work, in addition to the topics already outlined in this grade the following may be touched upon more in detail, namely: 1. Carpeting rooms. 2. Plastering and painting. 3. Board measure. 4. Stone and masonry. 5. Bins, wagon-boxes, tanks, and some knowledge of Discount as used in business.

The work should always be neatly done, properly punctuated, and ready to be set up in type if necessary to do so. The pupil must now do *more thinking* and less "*ciphering.*" Intelligent thought-work is the only work that pays. Rapid oral work induces thought. All dreamy, humdrum work should be avoided as a deadly mental poison.

Fifth Year's Work.

The preceding year's work took the pupil through what is usually called the "Elementary Arithmetic." In many graded schools the four years' course already outlined embraces five years; but this is arithmetic long drawn out. Beginning now with a Practical or Common School Arithmetic, the pupil has arrived at that stage in the development of his reasoning faculties when he must do more thinking and less mechanical ciphering. He can now take

hold of principles and apply them as he advances rapidly over the first half of the text, and parallel with the work in the Common School Arithmetic is that carried forward in the Mental Arithmetic through Fractions.

During this year all definitions should be learned and retained in the mind. Good definitions stand as "hitching-posts" to which knowledge must be tied.

To understand the full import of this remark, the teacher should take simple addition, after the class has passed over the subject in the Common School Arithmetic, and arrange an outline as follows:

Addition.

1. Definition.
2. Terms.....
   1. Parts.
   2. Sum, or Amount.
3. Signs......
   1. Plus.
   2. Equality.
4. Principles.
   1. Like numbers can be added.
   2. The sum is equal to all the parts.
   3. The sum is the same in kind as the parts.
   4. Units of the same order are added directly.
5. Operation.
   1. Writing the numbers.
   2. Drawing a line beneath, or making the sign of equality.
   3. Adding, reducing, etc.
6. Exercises.
7. Rule.
8. Proof.

Let a similar plan be followed after each subject is passed over, and the pupils will soon learn to arrange and classify quite accurately any subject that they study. This habit is invaluable, and it can be applied in other branches as advantageously as in Arithmetic. Such outlines, made by the pupils and corrected or modified by the teacher, afford excellent methods for reviews and analyses of topics. To pick up the scattered pieces of a subject and put them together into a consistent whole involves all the constructive and reflective faculties of the mind. To this must be added

the requisite skill in asking pointed, searching, and appropriate questions. The teacher who does not know how to ask questions at the proper time and in the right manner will never succeed in teaching any subject, much less Arithmetic. There should always be a plan in questioning either a pupil or a class, and that plan should usually be in an ascending scale. It is not enough that a pupil answers in a set form of words. The teacher must find out that the words used by the pupil are understood in their arrangement, and that they convey the correct idea to the learner's mind. What is said here has reference to relevant questions and not to that thoughtless jabber and clatter sometimes heard in recitations.

There are four things now that the pupil should learn: 1. To hear correctly the conditions of a problem, or to understand a problem when he reads it. 2. To remember exactly all the conditions when a problem is heard or read. 3. To think what is to be done. 4. To do it.

Let these steps be firmly impressed upon the learner's mind, and he will then use his faculties to much better advantage. As the knowledge in his mind is put into form, he is able to strengthen himself when he can pick out his own weaknesses. To grasp conditions with vigor, and to hold them tenaciously, are the first preliminary steps in the solution of a problem. These are followed by correct reasoning upon the data given. If a mistake is then made in the line of thought afterwards, the inference is that the pupil started in the wrong direction.

To find his mistake and to retrace his steps is one of the most important acts in the learner's progress. The manner of doing the work is a matter of no small consequence. Every one admires the beautiful clean page, free from all blemishes, whether it be a letter, a trial balance, a legal instrument, or what not. The same careful, exact work that pleases elsewhere should be secured in mathematics.

Since mathematics is an exact science, exact, neat, legible work is the only kind that should be accepted.

Solutions lay under contributon language and taste in the arrangement and presentation of each thought. Working questions for the sole purpose of "getting answers" is not worth much from the disciplinary side; hence all written work should be neatly, legibly, and carefully done. The mathematical signs abridge ordinary language, and they are to be employed as often as possible in the solution of problems.

Extent of the Year's Work.

This year's work in the Practical or Common School Arithmetic for an average class will comprise a thorough review of the fundamental rules, fractions, denominate numbers, mensuration of some plane surfaces and of some solids. Stress ought now to be put on the "why" at each step. The teacher should be ready to ask "why" whenever the pupil appears to be in doubt; or better still, if the pupil will ask himself the question. To secure good work, let the teacher call for a written solution of some problem, and then keep the neatest and the best as specimens to stimulate others to greater effort.

Teachers should avoid helping pupils and classes too much. Skillful questions by the teacher, not *class-carrying,* set the pupils to thinking, and are needed to arouse the latent energy of a class when all other efforts fail. Questions on each topic at first *ought to be aimed low,*—leveled to the learner's comprehension. Gradually they should take a higher and wider range. There is great danger in aiming too high at the beginning. The questions frequently go so far above the pupils' positive knowledge that vagueness and emptiness are succeeded by discouragement.

Definitions.

Each subject has its technical or arithmetical terms. These terms, if not already known, should be learned in connection with each topic as it is studied. The importance attaching to cancellation, divisors, and multiples depends upon two things: 1. To shorten work; 2. To test results. Consequently they are to be mastered as helps.

If a pupil is strong in fractions, he is well prepared to make proper advancement in his arithmetical studies; but till he masters fractions thoroughly, it is a waste of time for him to take up other topics. A good working knowledge is not sufficient. He must go to the root of the matter and master principles so thoroughly that when the teacher asks for the solution of a problem, it is given without hesitancy. Terms and processes must be understood, and thoroughly grounded in the mind. These objects are easily accomplished when one step is taken at a time.

The subdivisions to be learned and remembered are: 1. Definitions. 2. Classes of fractions as to kinds, as to value, as to form. 3. Terms, including numerator, denominator, similar, dissimilar. 4. Principles. 5. Reduction of fractions. 6. Applications under addition, subtraction, multiplication, division, etc. 7. The comparison of the treatment of fractions with the four fundamental rules.

Simple exercises are to be used at first for illustrating new subjects.

The connection between compound reduction and the reduction of fractions needs to be pointed out to the pupil; or in other words, he is better satisfied when he can trace relations himself.

Related knowledge can be hitched together, and the more hitching the pupil can do, the better he is satisfied with his own work. With the "why" at each step, there

should be another of "likeness" or "unlikeness" immediately following.

After common fractions, decimals should be studied with the same degree of thoroughness. Changing common fractions to decimals, and the reverse process, have a strong tendency in directing the pupil to observe the close connection existing between the two so-called species of fractions. Again, the reader is reminded that one problem worked out and well understood is worth pages of problems hastily sketched and partially comprehended.

For every problem selected from the book, let the pupil make a corresponding problem of his own. Encourage each pupil to make his own illustrations and methods of solutions. Spend considerable time in having pupils read and interpret problems without requiring solutions. This will test their knowledge in interpretation.

During this year's work the pupils should "clean up everything well as they go."

MENTAL ARITHMETIC.

A general direction for solving problems in Mental Arithmetic will be given in this connection, and it may be employed advantageously in the part of the Third Grade and through the Fourth Grade, and it should be continued for obvious reasons as long as the pupil studies Arithmetic.

Direction.—1. The teacher will read or state the problem once, slowly and distinctly. 2. The pupil, or class, will give the answer to the problem. 3. The pupil, or pupils separately, will give a short, connected, logical analysis, and a conclusion.

To be avoided.—1. Long, tedious analyses. 2. Letting the pupils use the text-book during recitations.

The laws of numbers are abstractions derived from ab-

stractions. Comparing an abstraction with another abstraction between which a relation can be affirmed or denied, brings out a conclusion clear and unquestionable to the pupil's mind. All such exercises develop the reasoning faculties if they appeal to the understanding. Mental Arithmetic, if properly taught, cultivates the reasoning faculties more than any other of the common-school branches. It requires the pupil to hold each question firmly in his mind while he puts the conditions together and derives a necessitated conclusion.

The Mental Arithmetic work during this period should be so arranged as to smooth the work in the Common School Arithmetic. In some cases it may very advantageously precede the written work. Each book should be studied separately and the lessons recited at different times, and if either must be slighted, let it be the Common School Arithmetic. A bright pupil will learn all that is necessary to be known in Mental Arithmetic to Percentage in four or five months; but too frequently the "holding-back and long-drawn-out policy" is adopted under the mistaken notion that there is great danger of impairing the pupil's thinking faculties. Active, vigorous thinking never hurts or wears out a man, woman, or child, if proper physical exercise alternates with it; but any amount of "rusting out" is induced by laziness. Let the teacher remember that Mental Arithmetic is to be pursued as a distinct and independent study. Never is it to be ciphered out, but it must be thought out. It is valuable to the pupil chiefly because it requires thought-work.

It is not every book labeled "Mental or Intellectual Arithmetic" that deserves the name. There are weak books that require no effort on the part of the pupil. Of course little benefit can be derived from the study of such books. A year devoted to Mental Arithmetic when a boy or a girl is properly prepared to take up the subject and to

push it vigorously is worth more than twice the time devoted to any other subject in the common-school course.

Pupils naturally prefer Mental Arithmetic to Written. It affords a better field for the display of intellectual skill and superiority. Each solution carries a conscious conviction that is a tonic to the reasoning powers.

In conducting recitations, quick, sharp, accurate work must be pursued. By gradual practice pupils will handle large numbers with almost as great ease as smaller numbers.

Sixth Year's Work.

Percentage and its applications, Ratio and Proportion, Evolution and Involution, the progressions, and mensuration, are the important subjects to be studied during this year. The same remarks in regard to mastering definitions, principles, and processes, made in the preceding year's work will apply to this grade. Percentage is simply a continuation of fractions. This idea is fundamental, and the sooner the pupil grasps it the more rapid headway he will make in the subject. To show the close connection between fractions and percentage is not a difficult matter.

For instance, $\frac{1}{5} = .2 = 20\%$. Here we have the same value expressed under three different forms, and in computing we may use any one we choose. 20% of anything not only means "per cent," but it means $\frac{20}{100}$, an abstract fractional number. The pupils should be so well drilled on the equivalent forms that one readily suggests the other. It is more convenient in practice to take $\frac{1}{6}$ of a number than to find $16\frac{2}{3}\%$ of it, and so of many other aliquot parts of 100.

Practically the solution of problems in Percentage involves three cases: 1. To find the Percentage. 2. To find the Rate. 3. To find the Base. The pupil must drill long enough on each case to master it perfectly before beginning the next one.

Following the usual notation, the three cases are represented thus:

$$(1)\quad B \times R\% = P.$$
$$(2)\quad P \div R\% = B.$$
$$(3)\quad P \div B \ = R\%.$$

A problem under each case will now be given and solved.

1. Find 40% of 960 sheep.

Solution. $100\% = 960$ sheep;
$1\% = 9.6$ sheep;
$40\% = 9.6 \times 40 = 384$ sheep. *Ans.*

Second Solution. $100\% = \frac{5}{5} = 960$ sheep;
$20\% = \frac{1}{5} = 192$ sheep;
$40\% = \frac{2}{5} = 192 \times 2 = 384$ sheep. *Ans.*

This problem in common language means that $\frac{2}{5}$ of 960 sheep are to be found. It is a good exercise to have the pupils change problems from the percentage to the common form.

The ciphering solution is given thus:

$960 \times .4 = 384$. *Ans.*, which corresponds to formula (1).

2. What per cent of \$80 is \$120?

Solution. \$120 is $\frac{3}{2}$ of \$80; but $\frac{3}{2}$ of any number is equal to 150% of that number; hence \$120 is 150% of \$80.

Or, $\frac{120}{80} = \frac{3}{2} = 150\%$. *Ans.* by formula (2).

3. A paid B \$20, which was $6\frac{1}{4}\%$ of what he owed B; what was the debt?

Solution. $6\frac{1}{4}\% = \$20;$
$1\% = \$\frac{16}{5};$
$100\% = \frac{(16 \times 100)}{5} = \$320.$ *Ans.*

Or, $\$20 \div \frac{1}{16} = \$320$; or, $\$20 \div .06\frac{1}{4} = \$320$. *Ans.*

If it becomes necessary to find the amount or difference,

the formulas are $B \pm P = A$ or D. A when the upper sign is taken, and D when the lower sign is used.

By extending the formulas to include the amount and difference, another case arises in practice. The following problem illustrates it:

A sold a horse for \$80 and gained $14\frac{2}{7}\%$ on the cost price; required the cost.

Solution.

$$100\% + 14\tfrac{2}{7}\% = \tfrac{8}{7} = \$80;$$
$$\tfrac{1}{7} = \$10;$$
$$\tfrac{7}{7} = \$70. \quad \textit{Ans.}$$

Or,

$$\$80 \div 1.14\tfrac{2}{7} = \$70. \quad \textit{Ans.}$$

Or,

$$100\% + 14\tfrac{2}{7}\% = 114\tfrac{2}{7}\% = \$80;$$
$$1\% = \$.7;$$
$$100\% = \$70. \quad \textit{Ans.}$$

If the pupil keeps his mind clear upon the cost price of an article, he is not likely to encounter any serious difficulty in understanding how to classify and solve problems in Percentage. Encourage pupils to work a problem in two or three different ways after they know how to do it one way well.

Commission and Brokerage.

This subject is not always clearly presented in the text-books. There are two distinct classes of problems, and confusion often arises in the minds of the pupils because they are unable to classify the problems properly. The two classes of problems are:

1. Those in which Commission is charged for *investing;* and

2. Those in which Commission is charged for *selling,* or for *collecting.*

The money handled by the agent in the business for the principal is the base. In the first class of problems the agent's commission should be deducted before he makes an

investment for the principal, and the agent should never charge commission on *his* commission.

The second class differs from the first in this: the agent charges commission on all he *sells* or *collects*.

From the foregoing it is evident that the agent's commission in the first class of problems is in the amount that is sent to him to be invested, and that he must deduct his commission before he invests the remainder for his principal. This gives rise to two processes, namely, the amount of commission the agent must receive, and the amount of money that the principal must transmit to include the agent's commission and the investment. Plainly, two transactions are involved:

1. The amount sent to the agent, to find how much he can invest, less his commission. (1)

2. The sum to be invested, to find how much must be sent to cover both commission and investment. (2)

Under the second class there are two kinds of problems, as follows:

1. The amount of sales, to find how much must be sent to the principal, less the agent's commission. (3)

2. The sum sent to the agent, to find the amount of the sale. (4)

Let the teacher select a problem under each of the preceding conditions, and then tell the pupils to classify them as (1), (2), (3), (4).

When they are familiarized sufficiently with the four kinds, let them begin with the solution of problems under each case. Commission should not be passed over until the pupils are able to discuss clearly and intelligently all ordinary problems in the text-books. If the distinctions are clearly marked and observed at the outset, Commission is easily understood; otherwise it is groping in the dark.

Interest.

As soon as the definitions are learned and the pupils have some knowledge of Interest, the teacher should borrow a statute from a Justice of the Peace, and let the class read the law on Interest and Usury. This brings forward to a certain extent the legal phase of the subject, and when promissory notes are spoken of, a good opportunity is afforded for teaching something of the law of contracts and the qualifications required of parties before they can legally make a contract.

Interest from the mathematical standpoint may be expressed under the most general form thus:

$$P \times R \times T = \text{Interest.}$$

Or, when expressed in words:

The Principal multiplied by the Rate expressed decimally, multiplied by the Time in years, equals the Interest.

While this is the general form, yet, in a large number of examples, the "6% method" possesses many advantages. It affords many excellent opportunities for analytical work, and lessens as well the labor in computation. Unless a problem is very simple, use the "6% method."

There are fewer mistakes when it is employed. To find the interest on $1 for any given time at 6% is always easy when the pupil *once knows how,* and then to find the interest on any number of dollars for the given time is a matter of simple multiplication. By all means perfect pupils in the "6% method." For rapid, accurate work it is certainly superior to any other method of computing interest employed. However, the pupil should be perfectly familiar with the general method and then, according to the nature of the problem, he can use whichever is more convenient.

Promissory Notes.

Under this head the following terms should be explained and illustrated: 1. *A Note;* 2. *A Negotiable Note, a Joint Note;* 3. *The Maker, or Drawer;* 4. *The Holder, or Payee;* 5. *Indorser;* 6. *Maturity;* 7. *Days of Grace;* 8. *Protest;* 9. *A Partial Payment;* 10. *Indorsement.*

In actual practice one case in Interest is chiefly used, and that is to find the amount due on a note or account; but in arithmetics all the cases arising from a consideration of the *Principal, Rate, Time, Interest,* and *Amount* are given. If any three of these five terms are given, the other two may be found.

Discount.

This is an obscure subject to the learner unless he gets the right start at the outset. The kind of *discount* that he studies most in his text-book is used very little in business transactions. To avoid confusion a correct *definition* of *Discount* should be given and illustrated.

Discount is a deduction from a price, or from a debt.

The kinds of Discount are three: 1. *True Discount;* 2. *Bank Discount;* 3. *Trade* or *Commercial Discount.* True Discount should be sharply contrasted with Bank Discount. In True Discount, the debt does not draw interest except in rare cases. If the debt, however, bear interest, then the amount should be the sum discounted. Make this point plain to the pupil. The teacher should draw two notes, one not bearing interest and the other bearing interest, and let the pupil find the present worth of each; then require the proof. Suitable questions will unfold the subject, and reveal its close connection with simple interest.

Bank Discount.

The clearest insight into Bank Discount is gained when pupils can go to a bank and see people depositing money

or drafts, checking out, receiving certificates of deposit, sending money to distant points by means of drafts and checks, discounting notes, getting bills of exchange, etc.

From almost any bank a teacher can get all the blanks necessary for illustrating the various transactions ordinarily occurring there. The nearer the approach to actual business, the greater the interest the pupils will take, and the better the knowledge they will have of the subject.

Here, again, contrast Bank Discount with True Discount. Bring out the agreements, and then the differences. Discount two notes of the same face value by each method. Let the notes bear interest, and then again take them as non-interest-bearing. Compare results.

The class should not leave Discount until it thoroughly comprehends the subject.

Trade Discount is so different from True or Bank Discount that confusion is not likely to arise. Yet it is well enough to call attention to it.

Clearness in teaching depends entirely upon the grasp the teacher has on the subject. Clear, sharp-cut knowledge—how to communicate it, and how to interest pupils in the subject, will insure success.

Insurance.

This subject presents no serious obstacles, yet the discussion in the Arithmetic can be greatly enlarged if the teacher will apply to "Insurance Agents" for blanks, circulars of information, and other printed matter.

Then, there are the different kinds of Insurance, as well as the different plans of Insurance Companies, all of which should be studied. The statutes of each State and the reports of the Insurance Commissioners will throw a great deal of light on this branch of business.

After a fire or cyclone, the teacher should explain how

the loss is adjusted, and, in case of a death, within what time the policy is paid. The true theory is to have the pupils study the subject as it is, not as it appears to be.

Stocks.

Begin the subject from the Daily Newspaper. Under the "Stock Exchange," or "Money and Markets," or "Bulls and Bears," have the pupils read the market reports and explain what they read. Such a report will most likely include Clearing House Statement, Total Transactions for the day or week, Bonds, Stocks, General and Local Markets. The newspaper is the mirror through which the pupils must read the volume of business as it is transacted at the great commercial centers. For every problem selected from the text-book, the pupils, or the teacher, should make two from the quotations given in the paper. The definitions used in Stock Exchange must be learned, or much of the language will be unintelligible ; also, the pupil needs to be reminded often of the difference between the *par value of stocks* and the *market value,* and on which of these *brokerage, assessments,* and *dividends* are computed. No new principle is involved in the solution of problems under this head.

Taxes.

This subject has a special interest because it reaches every home. Children, therefore, manifest more or less interest in those things which they hear discussed by their parents. For instance, it is a good thing for children to get some idea by whom and for what purposes taxes are assessed and collected, how they are used after collection, and many other questions that are incident to our system of revenue for national, State, county, municipal, township, and school-district purposes. A good opportunity is afforded for teaching a wholesome moral lesson on giving in the valuation of personal and real property to the assess-

or. When pupils know what is meant, or intended to be meant, in a problem, it is not hard for them to begin it understandingly.

Compound Interest and Foreign Exchange.

Compound interest is, if the intervals are many, tedious to compute; hence the Interest Tables found in many treatises. To these tables it is safer to refer when they are convenient. There is one difference only between simple interest and compound. Let the pupil mark it. **Interest can become principal by agreement between the contracting parties.**

Foreign Exchange does not differ in principle essentially from Inland or Domestic Exchange. Two or three points need emphasizing: 1. That a foreign bill is usually drawn in triplicate, called a set of exchange; 2. Simple Arbitration; 3. Compound Arbitration. A copy of a Bill of Exchange should be shown to the class; better still, secure a bill from a bank, and let the pupils examine it. Then require them to write bills. To illustrate Circular Exchange through different points, use cities in the United States as representative commercial centers. Let the teacher explain the history of banking as it has been developed within the last few hundred years, and particularly compare the financial condition of our country with what it was just before the Revolution, just after the Revolution, and what it is at present.

Ratio and Proportion.

Definite knowledge is demanded in the treatment of Ratio and Proportion. Give the pupil a clear notion of what Ratio is. It must be plain to him; otherwise he is "shadow-hunting." New terms are introduced, and these he must learn too. The definition of Proportion follows from that of Ratio. Drill the class in Simple Proportion

till all the members can state problems correctly every time. The statement is the important point. Show how a proportion may be expressed in different ways, and that all are correct. Illustration: $4 : 8 :: 6 : 12$; or $4 \div 8 = 6 \div 12$; or $\frac{4}{8} = \frac{6}{12}$; or $4 : 8 = 6 : 12$.

Explain the term Proportion. Submit many problems for the class to state in proper form.

As a mental discipline, it is a strengthening exercise to have pupils analyze all the problems that they solve by simple proportion. This will show, too, that the same result can be obtained in more ways than one. Aside from this, however, Proportion has a much wider application than this narrow view would seem to indicate, namely, its application in the other branches of mathematics.

Compound Proportion is best taught by *Cause and Effect*. At least, my experience in teaching large classes is that pupils will learn it in much less time, and will seldom or never make mistakes in statements. In short, I regard the ordinary rule for stating questions in Compound Proportion as so much *obsolete* matter, retained in the books on account of its respectable antiquity.

Practice a class upon easy exercises at first. Keep

1st cause : 2d cause :: 1st effect : 2d effect

on the board or slate to prevent mistakes and to test the work.

Later on the work may be written at once for cancellation thus: $\frac{1C.}{2E.}\Big|\frac{2C.}{1E.}$; and later still, $\frac{1C. \times 2E.}{2C. \times 1E.} = \textit{Answer}.$

Proportion is one of the easiest subjects in the Arithmetic when rightly presented. Many times it is made one of the most obscure.

Square Root and Cube Root.

There is no difficulty experienced in teaching pupils how to square, cube, or raise numbers to any required power.

The process is that of simple multiplication; but to *un*-multiply the numbers is the troublesome part of the work. Evolution, then, is *undoing* what has already been done. To show the reason for resolving a number into two, three, or any number of equal factors is now the duty of the teacher. This may be done in several ways, namely: 1. The arithmetical method. 2. The geometric method. 3. The algebraic method. For pupils studying arithmetic, a combination of the arithmetical and geometric methods is preferable for both square and cube root. To give directness to the instruction in square root, after squaring a number, while resolving it into two equal factors, the geometric forms should be used to illustrate and to impress each step; in the absence of such forms, diagrams can be drawn on the blackboard. But both teacher and pupils can "whittle out" pieces of shingle that will answer the purpose very well, or cut out pieces of paste-board. Slices of a raw potato, apple, or turnip in case "of a pinch" make a fair substitute. However, all schools should be furnished with a full set of geometric forms. Some excellent teachers prefer to have the pupils take a square of either wood or paper and make additions to it before attempting to extract the square root of a number consisting of two periods. This paves the way for illustrating the numerical problem most advantageously. In combining the arithmetical computation with the geometrical construction, the learner's understanding is most effectively and permanently reached. If necessary let the pupil make three or four additions to the original square. Such work will delight him if the number whose root is to be extracted is a large one. The area of the original square of each additional piece should be calculated separately.

To facilitate work in square and cube root, the pupils should learn all perfect squares to four places of figures. This is not a difficult feat when a little reflection will show

in what figure each perfect power must terminate, and how easy it is "to square" mentally any integral number from 1 to 99. Again, the simple method of forming a table of perfect squares by the addition of the successive odd numbers reveals one of the most beautiful laws in connection with figures. More of this farther on.

Cube Root should always be illustrated by using the blocks. First take a set of blocks for making one addition, preserving the form of a cube. Assume the edge of the cubical block as a definite number of inches, and then have its contents found. Next determine the length and thickness of each of the seven pieces that must be added to preserve the form, and then find the volume of each piece separately. This preliminary work must be continued till the pupils see why each step is taken, and can also explain the operation satisfactorily. When cube root is once rightly learned, it is never entirely forgotten. The pictorial illustrations in the text-books, and such other devices as suggest themselves to the inventive teacher, will render the acquisition of this subject quite easy for the pupil of moderate arithmetical aptitude.

I would not recommend the early introduction of the algebraic formula for the extraction of the cube root of numbers, although it may be introduced very appropriately farther on in the course. The safer method seems to be this: **Each problem in cube root is to find the edge of a cube whose root must be the length sought; or, given the volume of the cube, to find its edge.**

Series.

As much as can be attempted under Series in this chapter is the presentation of a few of the very simplest cases. These should be applied chiefly to practical, or supposed practical, problems. The difference between Arithmetical Progression and Geometrical Progression ought to be

broadly outlined, and the distinction between a constant difference and a constant multiplier fixed in the mind and rationally apprehended by the pupil. A deeper insight may also be gained if some problems are proposed and solved which are related to compound interest and annuities. Anything like a satisfactory presentation of Series must be sought in our most advanced treatises on Algebra.

Mensuration.

There are so many ways in which measuring surfaces and volumes are applied, and such constant reference to books in order to solve common questions, that it is surprising how lightly the subject is touched upon ordinarily. Every boy ought to know how to find the contents of a piece of timber, having the dimensions given; or to tell how many bushels of corn a wagon-bed holds, and what deductions to make if the corn is not shelled; the number of cubic yards in a cellar or cistern; the number of tons in a hay-rick or a hay-mow; the number of brick in a building or wall; the number of gallons of water that a well, cistern, barrel, tub, or bucket will hold. Then there are formulas and rules for finding the area of circles, the surfaces and volumes of spheres, pyramids, cones, etc., that must be learned. Instead of solving such problems with the books open before them, pupils should know how to solve such problems, and commit certain data to memory ready for use. Some things are to be learned and retained because they are necessary and useful. To encourage pupils to look beyond their arithmetics for the reasons involved in certain processes, the teacher can refer to the geometry in which demonstrations are found. The impetus given to the mind to look beyond the present boundary of knowledge is a powerful incentive, and when wisely directed leads to excellent results.

Miscellaneous Problems.

All miscellaneous problems should be solved by the pupil. They are usually put in the book to test the pupil's knowledge. They are also intended as a review, and a "rounding up" of the subject.

Outlines.

Let the pupil now make a complete outline of the subject as it is presented in his book. This will give him a comprehensive view of arithmetic, and it puts his knowledge into good form for wielding it with the least expenditure of vital energy.

Mental Arithmetic.

During this year the Mental Arithmetic should be thoroughly completed. Whether a class commenced the year's work with percentage or the miscellaneous problems preceding percentage, every difficult problem should be solved before beginning a new collection. Commencing the miscellaneous collection first, then the following kinds of problems will help to strengthen the reasoning faculties:

1. Given the sum or the difference of the parts of a number, increased or decreased by a certain number, to find the original number.

2. Given one part of a number of times another, or one part of a given number more than another, or the number of times one part equals a number of times another, or the part proportional to a given number, to find the required number.

3. Problems in Proportion.

The right kind of work in Mental Arithmetic now not only gives a mastery of the Arithmetic itself, but it also lays the foundation for future excellence in Algebra.

Under Percentage and Interest the same care in the solution of problems should be observed as was indicated in the work of the preceding year. Mental Arithmetic and the problems solved are not to be Written Arithmetic. Work without pen or pencil is demanded. No matter how complicated a problem appears to be, it must be investigated strictly upon the mental basis. The best results are thus obtained. No deviation from the plan outlined ought to be tolerated. Let all the cases in Percentage and Interest be mastered, and the pupil's progress in these rules ever after will be easy and rapid, however difficult the problems may appear. The analytic habits of mind thus inculcated will prove invaluable in other directions.

There are other problems, however, that demand considerable attention, such as the pasture problems, beggar and number problems, labor problems, fish problems, pursuit problems, stock problems, horse and saddle problems, gaming problems, time problems, age problems, step problems, partnership problems, involution and evolution problems, will problems, and miscellaneous problems. All these problems except the miscellaneous ones should be studied in classes; that is, all fish problems should be studied and solved consecutively. When commenced, the particular class should be completed before any other problems are taken up. In Mental Arithmetic all these special problems ought to be placed in groups or classes. Two American authors have adopted this idea in the classification of these problems—Dr. Brooks and Mr. George E. Seymour.

Problems have two phases; one the arithmetical, and the other the graphical. These two make the deepest and clearest impressions when combined. When the relations are thought out and expressed, the next step is for the pupil to draw a representation of the conditions as explicitly stated, and then to derive the implied conditions. As an illustration, suppose a fish problem is given for solu-

tion. After the problem is solved, then let each pupil draw a picture of a fish representing all the given conditions, and show from these how the required conditions are found. The illustrative process should not precede the solution. It is not necessary to illustrate every problem, but the process of illustrating should be so well understood that the pupils can devise illustrations whenever needed. Some pupils require graphic representations in order to seize all the conditions of a difficult problem, and in such cases the picture is an aid to the pupil.

One problem of a class well understood by the pupil prepares him admirably for solving all similar questions. But it is oftentimes necessary to vary from the typical problem to elucidate a certain feature which is not quite clear to the pupil's mind. Of these variations the teacher is the judge. As a general thing, if the pupil can solve the most difficult problems, he will take care of the easy ones. If the habit is once formed by the pupil of going back and coming up again when he finds a "tough customer" until he succeeds in solving it, his success in Mental Arithmetic is assured.

Advanced Arithmetic.

Under this head will be given problems, principles, and comments.

Rapid Methods of Adding.

1. To add from left to right.

$$\text{I.}\quad \begin{array}{r} 74 \\ 69 \\ \hline 143 \end{array}$$

Explanation: $60 + 70 + 13 = 143$. The eye is accustomed to pass from the left to the right; hence we say $60 + 70 = 130$, and $130 + 13 = 143$.

Or we can increase 69 to 70, then add, and subtract 1 from the sum.

This process of adding two or more numbers should always be performed mentally.

Let it be required to add a column consisting of 85 + 97 + 68 + 73 + 56. Adding the tens first, we have 350; next the units, 29; hence, 379.

2. To add columns in groups of 10, 20, 30, etc.

This process consists in placing a small figure on the margin of the column, indicating the number of units in excess in each group. Accountants use this method, because if stopped when adding a column the small digits along the margin indicate the work as far as the last group added. The same work is performed by those mentally who touch the thumb and forefinger when the first group is reached, the second finger at the second group, and the open hand at the fifth group, and so on, repeating. Groups of "20" appear most convenient in practice. "Practice makes perfect" is the motto.

3. To add two or more columns at the same time.

Here, again, it is most convenient to add from left to right. The addition is performed mentally.

Thus:

	96	
183	87	
	45	228
297	69	
	78	
	375	

Explanation. By a previous method 183 is found; 45 added gives 228; 69, 297; 78, 375. Or it may be done thus:

$90 + 80 = 170$; $170 + 13 = 183$; $183 + 40 = 223$;
$223 + 5 = 228$; $228 + 60 = 288$; $288 + 9 = 297$;
$297 + 70 = 367$; $367 + 8 = 375$.

The entire mental work is expressed in detail.

4. To add three columns proceed thus:

```
     463
     921
  13    84
```

Conceive the hundreds column to be separated from the other two columns; then add and combine.

5. To add four columns at once.

Conceive the two left-hand columns separated from the two right-hand columns; add the left-hand ones separately, then the right-hand, and combine the results. Thus:

```
     4862
     5978
     8321
  190    161
```

Combining we have 19161.

Some Contractions in Multiplication.

1. To multiply by 5. Annex a 0 to the multiplicand, and divide by 2.

2. To multiply by 15. Annex two 0's to the multiplicand, and to the result add half itself.

3. To multiply by 25. Annex two 0's, and divide by 4.

4. To multiply by 75. Annex two 0's, and take $\frac{1}{4}$ of the result; the remainder is the product.

5. To multiply by 125. Annex three 0's, and divide by 8.

6. To multiply by 175. Annex two 0's, multiply by 7, and divide by 4.

7. To multiply by 275. Annex two 0's, multiply by 11, and divide by 4.

8. To multiply by $166\frac{2}{3}$. Annex three 0's, and divide by 6.

9. To multiply by 250. Annex three 0's, and divide by 4.

10. To multiply by $333\frac{1}{3}$. Annex three 0's, and divide by 3.

11. To multiply by 375. Annex three 0's, multiply by 3, and divide by 8.

12. To multiply by 625. Annex three 0's, multiply by 5, and divide by 8.

13. To multiply by $666\frac{2}{3}$. Annex three 0's, multiply by 2, and divide by 3.

14. To multiply by 750. Annex three 0's, multiply by 3, and divide by 4.

15. To multiply by 875. Annex three 0's, multiply by 7, and divide by 8.

16. To multiply two numbers when the sum of their units equals 10 and their tens figures are alike.

Thus: 84×86. Multiply the units and write the result; multiply 8 by 9, and combine.

Operation. $6 \times 4 = 24$; $8 \times 9 = 72$; hence, 7224. *Ans.*

17. To multiply a number less than 100 by itself.

Thus: 75×75. Multiply the units, and write the result; then multiply 8×7, and combine.

Operation. $5 \times 5 = 25$; $8 \times 7 = 56$; hence, 5625. *Ans.*

When the number does not terminate in 5, proceed thus: 68×68. The difference between 68 and 70 is 2; then $2 \times 2 = 4$, and $6 \times 7 = 42$. Since the tens place is vacant in the first product, we write 4204. *Ans.*

18. To multiply when the tens figures differ by 1 and the sum of the units equals 10.

Thus: 86×94.

Operation. $90 \times 90 = 8100$, and $4 \times 4 = 16$; hence, $8100 - 16 = 8084$. *Ans.*

Always square the unit figure of the greater number and subtract.

19. To multiply in a single line two numbers such as 232×238.

Operation. $2 \times 8 = 16$; $23 \times 24 = 552$; whence, 55216. *Ans.*

20. To multiply any number of two figures by 11. Write the sum of the two figures between them. Thus: $54 \times 11 = 594$.

21. To square any number of 9's instantaneously. Write as many 9's less one as there are 9's in the given number, an 8, as many 0's as 9's in the product, and a 1. Thus: $(9999999)^2 = 99999980000001$. *Ans.*

22. To square fractions of the form $6\frac{1}{2}$, $7\frac{1}{2}$, $8\frac{1}{2}$, etc. Multiply the whole number by the next higher digit, and annex $\frac{1}{4}$. Thus: $(6\frac{1}{2})^2 = 7 \times 6 + \frac{1}{4} = 42\frac{1}{4}$.

23. To multiply two like numbers when the sum of their fractions equals 1. Thus: $8\frac{1}{3} \times 8\frac{2}{3} = 8 \times 9 + \frac{1}{3} \times \frac{2}{3} = 72\frac{2}{9}$. Again, $39\frac{2}{3} \times 39\frac{1}{3} = 39 \times 40 + \frac{1}{3} \times \frac{2}{3} = 1560\frac{2}{9}$.

24. To multiply any number, as $5\frac{1}{4}$, by itself. Thus: $5\frac{1}{2} \times 5 + \frac{1}{4} \times \frac{1}{4} = 27\frac{9}{16}$.

25. To multiply two numbers mentally. Thus: 327×58. Here $327 = 300 + 20 + 7$, and $58 = 50 + 8$.

Begin at the left.

$$
\begin{aligned}
300 \times 50 &= 15000\\
20 \times 50 &= 1000\\
7 \times 50 &= 350\\
300 \times 8 &= 2400\\
20 \times 8 &= 160\\
7 \times 8 &= 56\\
\hline
& 18966
\end{aligned}
$$

Separating numbers and multiplying as in the last example is a valuable exercise. Any one who will practice it a short time will be astonished at the ease with which such operations are performed mentally. The advantage consists in working with numbers which the mind easily remembers.

The following are some difficult arithmetic problems whose solutions will be very acceptable to a large class of readers.

1. A brewery is worth 4% less than a tannery, and the tannery 16% more than a boat; the owner of the boat has traded it for 75% of the brewery, losing thus $103; what is the tannery worth?

Solution. Let 100% = the value of the tannery.
Then 96% = " " " " brewery,
and $100 \div 116 = 86\frac{6}{29}\%$ = the value of the boat.
By the problem,

$$75\% \times 96\% = 72\% = 75\% \text{ of the brewery;}$$
$$\therefore 86\tfrac{6}{29}\% - 72\% = 14\tfrac{6}{29}\% \text{ loss;}$$
$$1\% = \$7.25;$$
$$100\% = \$725. \quad \textit{Ans.}$$

Another Solution. $\frac{24}{25}$ of the tannery is worth the brewery, and the boat is worth $\frac{100}{116} = \frac{25}{29}$ of the tannery.

But the brewery is worth $\frac{3}{4}$ of $\frac{24}{25} = \frac{18}{25}$ of the tannery. Now if $\frac{25}{29}$ of the tannery is traded for only $\frac{18}{25}$ of the tannery, the loss equals $\frac{25}{29} - \frac{18}{25} = \frac{103}{725}$. This loss is $103; hence $\frac{103}{725} = \$103$, and $\frac{1}{725} = \$1$; $\therefore \frac{725}{725} = \$1 \times 725 =$ $725. *Ans.*

2. My agent sold my flour at 4% commission; increasing the proceeds by $4.20, I ordered the purchase of wheat at 2% commission; after which, wheat declining $3\frac{1}{3}\%$, my whole loss was $5; what was the flour worth?

Solution. Commission on $4.20 = $.08\frac{4}{17}$, and $\$4.11\frac{13}{17}$ = amount invested in wheat. $\$4.11\frac{13}{17} \times 3\% = \$0.13\frac{37}{51}$ decline on wheat. Total loss on $4.20 = $\$.21\frac{49}{51}$; $\$5 - \$.21\frac{49}{51} =$ $\$4.78\frac{2}{51}$ = loss on flour.

Let 100% = value of flour, and 96% = proceeds. Then $96\% \times \frac{2}{102} = 1\frac{15}{17}\%$ = commission for buying wheat; $96\% - 1\frac{15}{17}\% = 94\frac{2}{17}\%$ = investment in wheat; and $94\frac{2}{17}\% \times 3\frac{1}{3}\% = 3\frac{7}{51}\%$ = loss on wheat. Total expense and loss on

wheat $= 4\% + 1\frac{15}{17}\% + 3\frac{7}{51}\% = 9\frac{1}{51}\% = \$4.78\frac{2}{51}$; $1\% = \$.53$; $100\% = \$53$. *Ans.*

3. W. T. Baird, through his broker, invested a certain sum of money in Philadelphia 6's at $115\frac{1}{2}\%$, and three times as much in Union Pacific 7's at $89\frac{1}{2}\%$, brokerage $\frac{1}{2}\%$ in both cases; how much was invested in each kind of stock if his annual income is \$9920?

Solution. Let 100% = face of the Philadelphia stock, and 116% = cost of the same; then 348% = cost value of Union Pacific stock; $348\% \div 90 = 3.86\frac{2}{3}\%$ = face value of the same. 6% = income of the former, and $27\frac{1}{15}\%$ = income on the latter; but $6\% + 27\frac{1}{15}\% = 33\frac{1}{15}\% = \$9920$; $1\% = \$300$, and $116\% = \$34{,}800$; whence $348\% = \$104{,}400$, cost of the latter.

4. The Mutual Fire Insurance Company insured a building and its stock for two thirds of its value, charging $1\frac{3}{4}\%$. The Union Insurance Company relieved them of one fourth of the risk, at $1\frac{1}{2}\%$. The building and stock being destroyed by fire, the Union lost \$49,000 less than the Mutual; what amount of money did the owners of the building and stock lose?

Solution. Let 100% = value of property; $66\frac{2}{3}\%$ = risk; $1\frac{3}{4}\% \times 66\frac{2}{3}\% = 1\frac{1}{6}\%$, premium on value of the property; $\frac{1}{4}$ of $66\frac{2}{3}\% = 16\frac{2}{3}\%$ = Union risk; $66\frac{2}{3}\% - 16\frac{2}{3}\% = 50\%$ = Mutual's risk; $1\frac{1}{2}\% \times 16\frac{2}{3}\% = \frac{1}{4}\%$ = Union premium; $1\frac{1}{6}\% - \frac{1}{4}\% = \frac{11}{12}\%$ = Mutual's premium; $50\% - \frac{11}{12}\%$ = Mutual's loss; $16\frac{2}{3}\% - \frac{1}{4}\% = 16\frac{5}{12}\%$ = Union's loss; $49\frac{1}{12}\% - 16\frac{5}{12}\% = 32\frac{2}{3}\% = \$49{,}000$, excess of Mutual's loss; $1\% = \$1500$, and $100\% = \$150{,}000$.

$$100\% - 66\frac{2}{3}\% + 1\frac{1}{6}\% = 34\frac{1}{2}\% = \text{owner's loss.}$$

$$\therefore\ 34\frac{1}{2}\% = \$1500 \times 34\frac{1}{2} = \$51{,}750. \quad \textit{Ans.}$$

5. A merchant gives his note, 10% from date, for \$2442.04; what sum paid annually will have discharged the whole at the end of 5 years?

Solution. The interest due on \$2442.04 at the end of the first year is \$244.204; the principal must be diminished regularly by payments which are 10% more on each dollar every year.

First year, ...	\$1.00
Second year, ...	\$1.10
Third year, ...	\$1.21
Fourth year, ...	\$1.331
Fifth year, ...	\$1.4641
Total paid, ...	\$6.1051

To find the amount of the first payment, divide \$2442.04 by \$6.1051 = \$400. But \$400 + \$244.204 = \$644.204, which = the amount to be paid each year.

6. Hiero's crown, sp. gr. $14\frac{5}{8}$, was of gold, sp. gr. $19\frac{1}{4}$, and silver, sp. gr. $10\frac{1}{2}$; it weighed $17\frac{1}{2}$ lbs.: how much gold was in it?

Solution. The crown displaced $\frac{8}{117}$ of its weight in water; gold $\frac{4}{77}$, and silver $\frac{2}{21}$. Hence we must compare the combining weights with the weight of the crown in water.

$$\frac{8}{117}\begin{array}{l}\left\{\dfrac{2}{21}\right\} \\ \left\{\dfrac{4}{77}\right\}\end{array} \quad \begin{array}{l}\left\{\dfrac{726}{27027}\right\} \\ \left\{\dfrac{444}{27027}\right\}\end{array} \text{ or } \quad \begin{array}{c}74 \\ 121 \\ \hline 195\end{array}$$

The combining ratios are $\frac{74}{195}$ and $\frac{121}{195}$;

or $$\frac{17.5 \times 74}{195} = 6\frac{25}{38} \text{ silver,}$$

and $$\frac{17.5 \times 121}{195} = 10\frac{67}{78} \text{ gold.}$$

7. A dealer in stock can buy 100 animals for \$400, at the following rates: calves, \$9; hogs, \$2; lambs, \$1. How many may he take of each kind?

Solution. It is evident that the mean average price is \$4; also, that to sell a calf for \$4 that is worth \$9 is a loss of \$5; while \$2 is gained on each hog, and \$3 on each sheep. Since the mean average price is \$4, it follows that the losses and gains must balance, and such numbers must be taken as will balance the differences in prices.

Operation.

	1	3	5		10
4	2	2		5	60
	9	5	3	2	30
			8	7	100

Explanation. The first column represents the prices, the second column the gains and losses, the third column balances the losses and gains when the highest and lowest priced animals are combined, and the fourth column when the calves and hogs are combined. The sum of the third and the fourth columns is 15; but 15 will not divide 100, hence we must find 8 times some number plus 7 times some number that will equal 100, or 8 times a number plus 7 times some other number = 100; whence $8 \times 2 + 7 \times 12 = 100$; therefore, multiplying the separate numbers in third column by 2, and the separate numbers in fourth column by 12, we get 10, 60, and 30 for one set of answers. If we take 9 instead of 2, and 4 instead of 12, we have 45, 20, 35.

8. A hall standing east and west is 46 feet by 22 feet, and $12\frac{1}{2}$ feet high; what is the length of the shortest path a fly can travel, by walls and floor, from a southeast lower corner to a northwest upper corner?

Solution. Turn the west wall down flat with the floor outward; this will form a rectangle $58\frac{1}{2}$ feet in length and 22 feet in width, whose diagonal $= \sqrt{(58\frac{1}{2})^2 + (22)^2} =$ 62 + feet. Turning it down to the north side of the floor, the sides of the rectangle are $22 + 12\frac{1}{2} = 34\frac{1}{2}$ feet and 46 feet, whose diagonal $= \sqrt{(34\frac{1}{2})^2 + (46)^2} = 57\frac{1}{2}$ feet. *Ans.*

9. How many stakes can be driven down upon a space 15 feet square, allowing no two to be nearer each other than $1\frac{1}{2}$ feet; and how many allowing no two to be nearer than $1\frac{1}{4}$ feet?

Solution. 1. It is evident that eleven rows of stakes can be set "in squares," having 11 stakes in each row; but it remains to be shown that this is not the greatest number of stakes that can be set on the plat of ground. If we put 11 stakes on the bottom margin or first row, and if we place directly above the middle point of the distance between any two consecutive stakes a point at the required distance from the two points, forming an equilateral triangle, the second row will contain just 10 stakes, and the distance between the first row and the second row = $\sqrt{(\frac{3}{2})^2 - (\frac{3}{4})^2} = 1.29903 +$ feet. Hence the gain in width is $1.5 - 1.299903 + = .20097$ feet, and to gain an additional row seven rows must be set. Of the first nine rows, the first, third, fifth, seventh, and ninth will have 11 stakes each, and the second, fourth, sixth, and eighth, 10 stakes each. In the three remaining rows nothing can be gained, so these will have 11 stakes each. Therefore, $11 \times 8 = 88$, and $10 \times 4 = 40$, and the greatest number is 128. *Ans.*

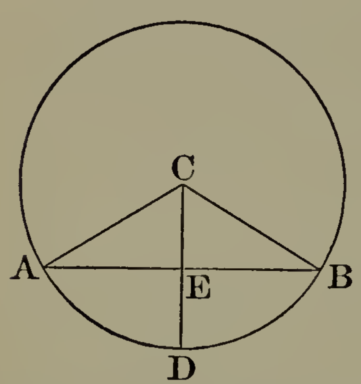
2. Arranging in squares, 169 can be set. Proceeding in the first part of the problem, we have five rows of 13, four

rows of 12, and five rows of 13 stakes, making a total of 178 stakes. If this is the greatest number, then the problem is solved with fourteen rows. Let us see if fifteen rows can be set. Now make the rows from the base line alternately reach the left side, and the second set of lines alternately reach the right side, and fifteen rows of 12 stakes in each row can be set $= 15 \times 12 = 180$ stakes.

Remark. This last method is regular "lattice-work."

10. A wooden wheel of uniform thickness, 4 feet in diameter, stands in mud 1 foot deep; what fraction of the wheel is out of the mud?

Solution. In the annexed diagram let $AC = BC = DC = 2$, the radius of the wheel; $CE = DE = 1$; then $AE = \sqrt{3} = 1.732 +$; $AB = 2\sqrt{3} = 3.464 +$ feet. By the rule given, for finding the area of the segment of a circle, the area of the segment in this case $= (1^3 \div 6.928) + \frac{2}{3}(3.464 \times 1) = .1443 + 2.3094 = 2.4537$. But the area of the wheel $= 12.5664$, and the area above the mud $= 12.5664 - 2.4537 = 10.1127$; whence $10.1127 \div 12.5664 = .804 +$ of the wheel above the mud.



11. By discounting a note at 20% per annum, I get $22\frac{1}{2}\%$ per annum interest; how long does the note run?

Solution. Discount = 20% of the face = $22\frac{1}{2}\%$ of the proceeds; proceeds $= \dfrac{20}{22\frac{1}{2}} = \frac{8}{9}$ of the face; discount $= \frac{1}{9}$ of the face $= 11\frac{1}{9}\%$; time $= \dfrac{11\frac{1}{9}}{20} = \frac{5}{9}$ year $= 200$ days.

12. A 12-inch ball is in the corner where walls and floor are at right angles; what must be the diameter of another

ball which can touch that ball while both touch the same floor and the same walls?

Solution. There are evidently two problems in this one: (1) When the ball is in the corner behind the 12-inch ball; (2) When it is in front of the 12-inch ball.

1. The distance from the corner of the room to the center of the 12-inch ball is the diagonal of a cube whose edge is 6 inches, i.e. the diagonal $= \sqrt{(6^2+6^2+6^2)} = 6\sqrt{3}$; the distance from the nearest point of the ball to the corner $= 6\sqrt{3} - 3$, and from the farthest point $= 6\sqrt{3} + 6$. Now assume the radius of the required ball to be 1; then the point at which it touches the given ball is $\sqrt{3} + 1$. By the problem this distance is $6\sqrt{3} - 6$; hence $(6\sqrt{3} - 6) \div (\sqrt{3} + 1) = 6(2 - \sqrt{3}) = 1.6077 +$ = radius of required ball, and its diameter = 3.2154 inches.

2. In this case we have $(6\sqrt{3} + 6) \div (\sqrt{3} - 1) = 6(2 + \sqrt{3}) = 22.3923$ inches, and the diameter = 44.7846 inches.

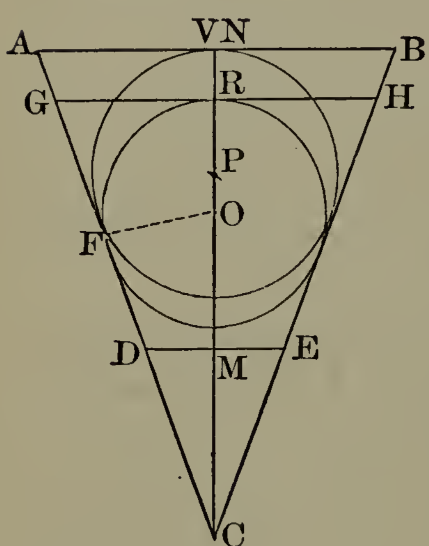
13. A workman had a squared log twice as long as wide or deep; he made out of it a water-trough, of sides, ends, and bottom each 3 inches thick, and having 11772 solid inches; what is the capacity of it in gallons?

Solution. Cut the log into two equal cubes The hollow part of each of these cubes = 5886 cubic inches. A cubical box having sides 3 inches thick may be regarded as made up of 6 square blocks, 12 blocks of the same length, and 3 inches wide, and 8 corner cubes of 27 inches each. Take away two of the square blocks and one long block, and the half-trough is left, containing 4 square blocks, 11 long blocks, and 8 cubical blocks, making a total volume of 5670 cubic inches. Now if we take a side surface in each, there are 4 squares and 11 strips = 1890 inches in area.

One square + $\frac{11}{4}$ strip 3 inches wide = $472\frac{1}{2}$ square inches. Divide the strip $\frac{11}{4}$ of 3 inches wide lengthwise into 2 strips each $4\frac{1}{8}$ inches wide and placed on adjacent sides of the square, and there will still be lacking a square $(4\frac{1}{8})^2 = 17\frac{1}{64}$ square inches. Therefore the completed square must contain $472\frac{1}{2}$ square inches + $17\frac{1}{64}$ square inches = $489\frac{33}{64}$ square inches; but $\sqrt{489\frac{33}{64}} = 22\frac{1}{8}$ inches. Consequently, $22\frac{1}{8} - 4\frac{1}{8} = 18$ inches, which is 6 inches less than the thickness of the log. Hence the log is 48 inches long and 24 inches wide. The volume of the trough = $(2 \times 18 \times 18 \times 21) \div 231 = 68\frac{8}{11}$ gallons.

14. A tin vessel having a circular mouth 9 inches in diameter, a bottom $4\frac{1}{2}$ inches in diameter, and a depth of 10 inches is $\frac{1}{4}$ part full of water; what is the diameter of a ball which can be put in and just be covered by water?

Solution. Let the annexed figure represent a middle section of the vessel, O being the center of the ball which is just covered with water, and P being the center of a ball that will just drop into the vessel and is level with the top of the vessel; the line DE represents the lower part of the vessel. Conceive the vessel to be extended to C, forming a regular cone. By the problem, $AN = BN = 4\frac{1}{2}$ inches; NM = 10 inches = MC. Then $AC = \sqrt{20^2 + (4\frac{1}{2})^2} = 20.5$ inches. The area of the triangle ABC = 90 square inches, and its perimeter = 50 inches. It is shown in geometry that the radius of an inscribed circle = the area of the circumscribing triangle divided by half the



perimeter; therefore the radius $NP = 90 \div 25 = 3.6$ inches.

If GRH represents the surface of the water after the sphere, O, is dropped into it, there are given the cone NC and its inscribed sphere, P, and its radius, NP, to find the radius FO of the sphere inscribed in the cone RC. This involves the principle of similar solids.

$$\text{Volume of cone } ABC = (4\tfrac{1}{2})^2 \times \tfrac{20}{3} \times \pi = 135\pi;$$
$$\text{Volume of sphere } NP = (7\tfrac{1}{5})^3 \times \tfrac{1}{6}\pi = 62.208\pi.$$

The ratio of the sphere to the cone $= \frac{288}{625}$, if we consider the cone as 1, then the volume of the cone $DEC = (\frac{1}{2})^3 = \frac{1}{8}$, since MC is half NC. The volume of $ABDE = \frac{7}{8}$, and $\pi = 3.1415926$.

The volume of water is $\frac{1}{4}$ of $\frac{7}{8} = \frac{7}{32}$, but the volume of the cone $ABC -$ the volume of the sphere $NP = 1 - \frac{288}{625} = \frac{337}{625}$, and the volume of the cone $GHC -$ the volume of the sphere $FO = \frac{7}{8}$ of cone $DEC + \frac{7}{32}$ of the water $= \frac{11}{32}$.

Similar solids are to each other as the cubes of their like dimensions; hence

$$\tfrac{337}{625} : \tfrac{11}{32} :: (3.6)^3 : (3.09836)^3 = (FO)^3,$$

whence $3.09836 \times 2 = 6.19672$ inches, required radius.

15. How many inch balls can be put in a box which measures, inside, 10 inches square and is 5 inches deep?

History. This problem is due to Dr. U. Jesse Knisely, whose solution created no little controversy owing to his method, which was different from that given by all preceding arithmeticians. With his solution discussion began. As the problem was more carefully examined, other solutions were published.

The appended solutions indicate the progress of the investigation.

This is one of the most remarkable arithmetical problems ever published.

Dr. Knisely's Solution is as follows: Place on the bottom 100 balls; above these, 81 balls; then 100, then 81, and so on as high as possible without entering the fifth inch of the height. The top of the second and each subsequent layer so placed is .7071 inch higher than the preceding. Without passing into the fifth inch there can be, thus placed, $100 + 81 + 100 + 81 + 100 = 462$ balls, and occupying 3.828 inches of the depth; there remains a space 10 inches square and more than 1 inch deep. How many balls can be placed in this space? Starting with a row of 10, and arranging triangularly, so as to have rows of 10 and 9 alternately, the second and each succeeding row will extend $\frac{1}{2}\sqrt{3}$, or .866 inch farther than the preceding; hence there could be eleven rows, six 10's and five 9's, or 105.

But if we let the triangular arrangement cease with the ninth row, there will be five rows of 10 and four rows of 9, making 86 balls, occupying 1 inch. $.866 \text{ inch} \times 8 = 7.92$ inches of the width; there remains a width of 2.07 inches, which will allow two rows of 10. Then, $86 + 20 = 106$, and $462 + 106 = 568$. *Ans.*

Second Solution, by Mr F. W. Brown, Byer, Ohio. He places four rows of 9 balls each on the 5×10 inches base of box, having the outside rows against the sides and a space of $1 \div 3 = \frac{1}{3}$ of an inch between two contiguous balls across the box and $1 \div 8 = \frac{1}{8}$ of an inch lengthwise. This makes 36 balls in bottom layer. On this he places three rows of 8 balls each, so that one touches 4 balls in bottom layer. By a calculation he makes the balls of the second layer extend just below the plane passing through the center of the balls of the first layer; hence the second layer extends upward not quite half an inch above the top of the first layer. The third layer is placed directly

on top of the first, and the fourth on the second, and so on till nineteen layers are thus put into the box.

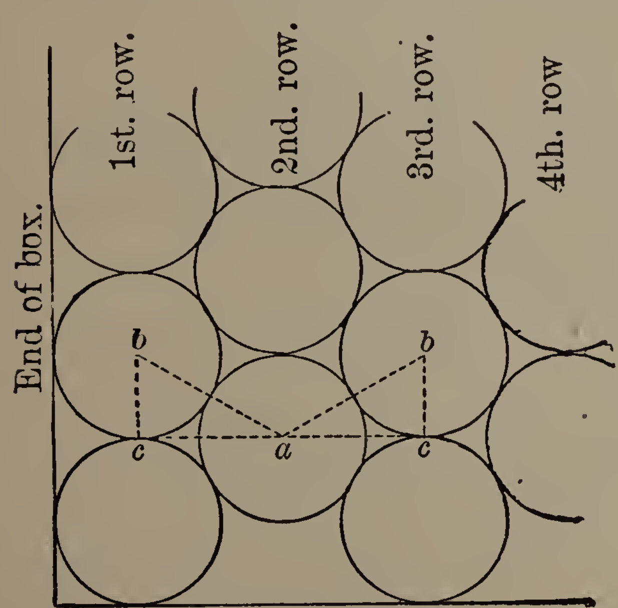
The first layer is repeated nine times and the second eight times, giving a total of

$$\left\{\begin{array}{c}(4 \times 9) \times 10 = 360 \\ \text{and} \\ (3 \times 8) \times \;\; 9 = 216\end{array}\right\} 576 \text{ balls.}$$

Third Solution, by Mr. F. F. Vale of Ohio. He puts in 100 on the 10×10 base; then 81 in second layer, 100 in third, 81 in fourth, and 100 in fifth, making 462 balls. The sixth and seventh layers are placed in such a way that five rows of 9 balls and one row of 10 balls are in the sixth, or 55 balls, and six rows of 10 balls, or 60 balls, in the seventh. He makes the centers in fifth layer $4 \times .797107 = 2.828428$ inches above centers in first layer, and $4 - 2.828428 = 1.17157$ below centers of the seventh layer.

Hence the total number is $462 + 115 = 577$. *Ans.*

The Fourth Solution was obtained by Mr. A. J. Trapp, Pleasant Hill, Missouri. If the box were but 1 inch deep, it manifestly would hold only 50 balls. These would just fill the box if arranged in ten rows of 5 balls each, the centers of adjacent rows being 1 inch apart. But

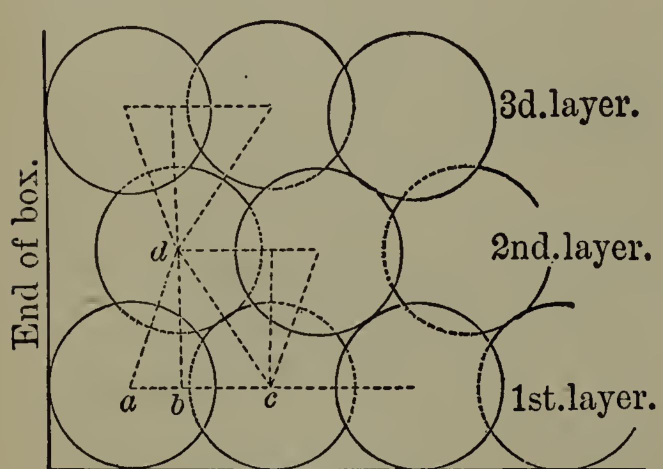


Side of box.
Fig. 1. Plan of 1st layer.

$ab = 1$ inch;
$cb = \frac{1}{2}$ inch;
$\sqrt{1^2 - \frac{1}{2}^2} = .86603 -$

if the balls in the *even* rows (second, fourth, sixth, etc.) are placed in the cavities between balls in the odd rows (first, third, fifth, etc.), the centers of adjacent rows will be only .86603 — inch apart, and eleven rows can be placed in the box, the odd rows having 5, and the even rows 4 balls, and an unoccupied space of .339 + inch will be left between the outside of the eleventh row and the end of the box; that is, the 50 balls so arranged occupy a space 1 inch high, 5 inches wide, and 9.661 inches long, their aggregate volume filling $\frac{54}{100}$ of this space. This is evidently the most compact arrangement possible for a single layer of balls.

Now as to the most compact arrangement for two or more layers. If a ball were placed on top of each ball in the first layer, the two layers would contain 100 balls, and would occupy a space $2 \times 5 \times 9.661$ inches; the aggregate volume of balls filling $\frac{54}{100}$ of the total volume of space as before. If, however, the balls in the second layer are placed in the trihedral pits between the adjacent rows in the first layer, the plane passing through the centers of balls in the second layer will be but .8165 inch above a similar plane in the first layer, and the two layers will go into a box 1.8165 inches high.



Bottom of box.
Fig. 2.

$cd = 1$ inch;

$ac = ad = .86603$ inch.

area $acd = .353556 +$.

and

$$\frac{.353556+}{\frac{1}{2}\,ac} = db = .8165-$$

also,

$$\sqrt{ad^2 - db^2} = ab = .289-.$$

Since the centers of the rows in the second layer are .289 inch to the right of the center of the corresponding row

in the first layer, and since the outside of the eleventh row in the first layer is .339 inch from the end of the box, there is $.339 - .289 = .05$ inch more than enough room in the box for eleven rows in the second layer; that is, the two layers occupy a space 1.8165 inch high, 5 inches wide, and $10 - .05 = 9.95$ inches long. But the second layer contains only 49 balls—the six odd rows containing 4 and the five even rows 5 balls each; therefore the aggregate volume of the 99 balls in the two layers fills more than $\frac{57}{100}$ of the volume of the space occupied. Hence this is the most compact possible arrangement of layers, the loss of one ball in the even layers being more than compensated by the gain in space caused by decreasing the height between all the layers to .8165 inch.

Since the depth of the box is 10 inches, it will contain twelve layers $\left(\frac{10 - 1}{.8165} + 1\right)$, leaving .0185 inch to spare above the upper layer.

The six odd layers contain			50 balls each			= 300
The six even	"	"	49	"	"	= 294
Total number of balls in box,						594

The aggregate volume of the 594 balls fills $\frac{622}{1000}$ of the total volume of the box. The same number of balls could be placed in a box $5 \times 9.95 \times 9.9815$ inches, and would occupy $\frac{626}{1000}$ of its volume.

16. My tailor informs me it will take $10\frac{1}{4}$ square yards of cloth to make me a full suit of clothes. The cloth that I am about to purchase is $1\frac{2}{3}$ yards wide, and on sponging it will shrink $\frac{1}{20}$ in width and length. How many yards of the above cloth must I purchase for my "new suit"?

Solution. The author designed this problem to be solved by proportion. Whence

20 : 19 : : 1.875 yds. : 1.78125 yds. = width after sponging.

Again,

20 : 19 : : 1.78125 yds. : 1.6921875 yds. = second shrinkage.

Consequently, 1.6921875 yds. : 1 :: 10.25 : $6\frac{62}{1083}$ yards.
Ans.

17. Suppose a clock to have an hour-hand, a minute-hand, and a second-hand, all turning on the same center. At 12 o'clock all the hands are together and point at 12.

(1) How long will it be before the second-hand will be between the other two hands and at equal distances from each?

(2) Also before the minute-hand will be equally distant between the other two hands?

(3) Also before the hour-hand will be equally distant between the other two hands?

Solution. Let the annexed diagrams represent the posi-

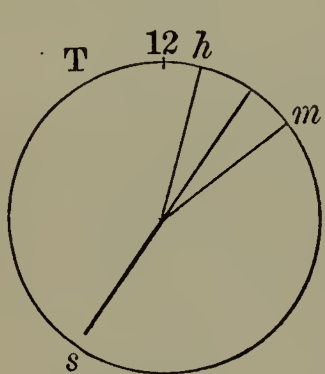


Fig. 1

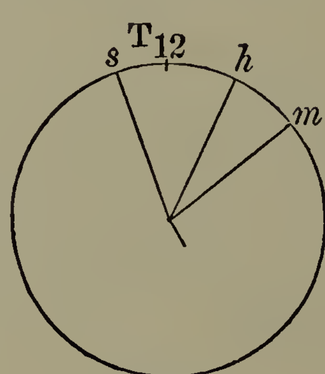


Fig. 3

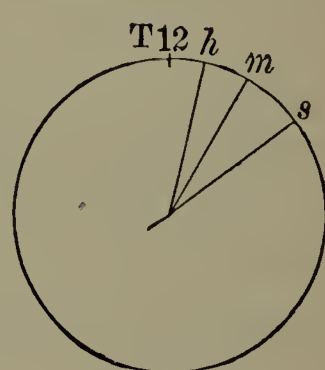


Fig. 2

tions of the three hands respectively when they first satisfy the conditions of the question. When the hour-hand moves over 1 space, the minute-hand moves over 12 spaces and the second-hand moves over 720 spaces.

1. From T to $h = 1$, T to $M = 12$, T to $S = 720$, M to $S = 708$, S to $h = 708$, h to $M = 11$. Hence the number of spaces around the face of the clock $= 708 + 708 + 11 = 1427$, and the second-hand has passed over $\frac{720}{1427}$ of the distance, and the time equals $\frac{720}{1427} \times 60 = 30\frac{390}{1427}$ seconds after 12 o'clock, and it will be equally distant between the two hands again at $\frac{720}{1427} \times 60 = 60\frac{780}{1427}$ seconds after 12 o'clock. This last answer is the one Greenleaf gave.

2. In figure 2, T to $h = 1$, T to $M = 12$, and once around plus T to $S = 720$ spaces; h to $M = 11$, M to $S = 11$; hence the number of spaces around the face of the clock is 697. Therefore the time $= \frac{720}{697} \times 60 = 61\frac{683}{697}$ seconds after 12 o'clock.

3. In figure 3, T to $h = 1$, T to $M = 12$, h to $M = 11$, T to $S = 720$, counting around the face, T to $h = 11$; hence the spaces once around the face $= 720 + 10 = 730$, and the time $= \frac{720}{730} \times 60 = 59\frac{13}{73}$ seconds after 12 o'clock.

18. From a cask containing 10 gallons of wine, a servant draws off a gallon each day for twenty days, each time supplying the deficiency by the addition of a gallon of water; and then, to escape detection, he again draws off 20 gallons, supplying the deficiency each time by a gallon of wine. How much water still remains in the cask?

Solution. After each day's drawing from the cask $\frac{9}{10}$ of its previous contents will be left. The quantity of wine left the first day is $\frac{9}{10}$ of 10 gallons; second day, $\frac{9^2}{10^2}$ of 10 gallons; and at the close of the twentieth day $= \frac{9^{20}}{10^{20}} \times 10$. Now, the quantity of water in the cask at the end of the second twenty days would be equal to the quantity $\left(\frac{9^{20}}{10^{20}} \times 10\right)$ multiplied by $\frac{9^{20}}{10^{20}}$.

$$.9^{20} = .12157665459;$$

and

$$10 - .12157665459 \times 10 = 8.7842334541 \text{ gallons,}$$

the quantity of water in the cask at the end of twenty days; hence

$$8.7842334541 \times .9^{20} = 1.0679577$$

gallons of water still in the cask.

19. James Page has a circular garden, 10 rods in diameter; how many trees can he set in it so that no two shall be within 10 feet of each other, and no tree within $2\frac{1}{2}$ feet of the fence inclosing the garden?

Solution. Deduct 5 feet from the diameter and it leaves 160 feet. Planting 1 tree at the center of the circle, next set 6 other trees around it 10 feet from each other and from the center tree. These 6 trees form the six corners of a regular hexagon. Now plant another hexagonal row of trees around the first row at the required distance and it will contain 18 trees, 4 trees in each side of the second hexagon. Repeating this process, eight rows can be set around the center tree, until the eighth row contains 48 trees. Since the first hexagon contains 6 trees and the eighth 48 trees, the total number $= (6 + 48) \times 4 = 216$.

Counting the tree at the center, we have 217. But the sides of the eighth hexagon do not occupy all the space that can be used. Between each side and the circumference there is room for 4 more trees, or 24 trees additional; therefore the total number of trees is $217 + 24 = 241$.

Remark. This problem is easily solved by planting 17 trees on the diameter passing through the center of the garden, and then on the chords $\sqrt{75}$ feet apart, parallel to the first diameter. The trees are planted in the quincunx order.

20. A, B, C, D, and E play together on this condition: that he who loses shall give to all the rest as much as they already have. First A loses, then B, then C, then D, and last also E. All lose in turn, and yet, at the end of the fifth game, they have all the same sum, viz., each $32. How much had each before they began to play?

Solution. The easiest way is to reverse the work, as follows:

A	B	C	D	E	
$32	$32	$32	$32	$32,	what each had when they quit.
16	16	16	16	96,	end of fourth game.
8	8	8	88	48,	end of third game.
4	4	84	44	24,	end of second game.
2	82	42	22	12,	end of first game.
81	41	21	11	6.	*Answer.*

21. Seven men purchase a grindstone of 60 inches in diameter. What part of the diameter must each grind off so as to have one seventh of the whole stone?

Solution. There remains six sevenths of the stone after the first has ground off his part.

$$\text{1st diameter} = 60\sqrt{\tfrac{6}{7}} = \tfrac{60}{7}\sqrt{42} = 55.54921 \text{ inches};$$
$$\text{2d} \quad \text{"} \quad = 60\sqrt{\tfrac{5}{7}} = \tfrac{60}{7}\sqrt{35} = 50.70925 \quad \text{"}$$
$$\text{3d} \quad \text{"} \quad = 60\sqrt{\tfrac{4}{7}} = \tfrac{60}{7}\sqrt{28} = 45.35574 \quad \text{"}$$
$$\text{4th} \quad \text{"} \quad = 60\sqrt{\tfrac{3}{7}} = \tfrac{60}{7}\sqrt{21} = 39.27922 \quad \text{"}$$
$$\text{5th} \quad \text{"} \quad = 60\sqrt{\tfrac{2}{7}} = \tfrac{60}{7}\sqrt{14} = 32.07135 \quad \text{"}$$
$$\text{6th} \quad \text{"} \quad = 60\sqrt{\tfrac{1}{7}} = \tfrac{60}{7}\sqrt{7} = 22.67787 \quad \text{"}$$

Subtracting the first from 60, and the second from the first, and so on, the answers are 4.45079, 4.83996, 5.35351, 6.07652, 9.39348, and 22.67787 inches respectively.

22. Four ladies purchased a ball of exceedingly fine thread, 3 inches in diameter. What portion of the diameter must each wind off so as to share of the thread equally?

Solution. Three fourths of the ball remained after the first unwound her share.

$$\text{1st diameter} = 3\sqrt[3]{\tfrac{3}{4}} = \tfrac{3}{2}\sqrt[3]{6} = 2.72568 \text{ inches;}$$

$$\text{2d} \quad \text{"} \quad = 3\sqrt[3]{} = \tfrac{2}{4}\sqrt[3]{4} = 2.38110 \quad \text{"}$$

$$\text{3d} \quad \text{"} \quad = 3\sqrt{\tfrac{1}{4}} = \tfrac{3}{2}\sqrt[3]{2} = 1.88988 \quad \text{"}$$

Hence the

First	wound off	3.00000 − 2.72568	= 0.27432	inches;
Second	" "	2.72568 − 2.38110	= 0.34458	"
Third	" "	2.38110 − 1.88988	= 0.49122	"
Fourth	" "	1.88988	= 1.88988	"

23. Find what each of the four persons, A, B, C, and D, are worth, by knowing—

1st. That A's money together with $\frac{1}{3}$ of B's, C's, and D's is equal to $137.

2d. That B's money together with $\frac{1}{4}$ of A's, C's, and D's is equal to $137.

3d. That C's money together with $\frac{1}{5}$ of A's, B's, and D's is equal to $137.

4th. That D's money together with $\frac{1}{6}$ of A's, B's, and C's is equal to $137.

Solution.

$\frac{1}{3}$ (B's + C's + D's) = $\frac{1}{3}$ (B's + C's + D's + A's) − $\frac{1}{3}$ A's;

that is,

A's + $\frac{1}{3}$ (A's + B's + C's + D's) − $\frac{1}{3}$ A's = \$137;

hence

$\frac{2}{3}$ A's = \$137 − $\frac{1}{3}$ of the *sum of all.*

Or,

A's = $\frac{3}{2}$ of \$137 − $\frac{1}{2}$ of the *sum of all.*

B's = $\frac{4}{3}$ of \$137 − $\frac{1}{3}$ of the *sum of all.*

C's = $\frac{5}{4}$ of \$137 − $\frac{1}{4}$ of the *sum of all.*

D's = $\frac{6}{5}$ of \$137 − $\frac{1}{5}$ of the *sum of all.*

Adding the values of A, B, C, and D, we have the *sum of all* =

$(\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5})$ of \$137 $-$ $(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5})$ of the *sum of all.*
Or,

$(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5})$ of *sum of all* $=(\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5})$ of \$137;
whence the *sum of all* =

$$\frac{\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}} \text{ of } \$137 = \tfrac{317}{137} \text{ of } \$137 = \$317.$$

Substituting this value of the *sum of all,*

A = \$47, B = \$77, C = \$92, D = \$101.

24. Suppose from an acorn there shoots up a single stalk, at the end of the year; that, at the end of each year thereafter, this stalk puts forth as many new branches as it is years old; also suppose all the branches to follow the same law, that is, to produce as many new branches as they are years old. How many branches will this oak tree consist of at the end of twenty years?

Solution. At the end of the first year there will be one stalk, which denote by 1_0; at the end of the second year, $1+1_0$; at the end of the third year the branches, $1+1_0$, will become 1_2+1; the first, being two years of age, will produce two new branches, the other will produce one new one; these can be represented by $1_2+1_1+3_0$, the small figures denoting the age in years of the stalks or branches. The results are summarized thus:

End of	1st	year,	1_0	= 1.
"	2d	"	1_1+1_0	= 2.
"	3d	"	$1_2+1_1+3_0$	= 5.
"	4th	"	$1_3+1_2+3_1+8_0$	= 13.
"	5th	"	$1_4+1_3+3_2+8_1+21_0$	= 34.
"	6th	"	$1_5+1_4+3_3+8_2+21_1+55_0$	= 89.

The law of the series is, **that twice any term, increased by the sum of all the preceding terms, gives the next term; or three times any term, diminished by the next preceding term, will give the next term.**

The terms are easily found by continual additions according to the law of the series.

0....New branches 1st year.
Total branches 1st year..............1
1.... " " 2d "
" " 2d " 2
3.... " " 3d "
" " 3d " 5
8 ... " " 4th "
" " 4th " 13
21.... " " 5th "
" " 5th " 34
55 ... " " 6th "
" " 6th " 89
144.... " " 7th "
" " 7th " 233
377.... " " 8th "
" " 8th " 610
987.... " " 9th "
" " 9th " 1,597
2,584.... " " 10th "
" " 10th " 4,181
6,765.... " " 11th "
" " 11th " 10,946
17,711.... " " 12th "
" " 12th " 28,657
46,368.... " " 13th "
" " 13th " 75,025
121,393 .... " " 14th "
" " 14th " 196.418
317,811.... " " 15th "
" " 15th " 514,229
832,040.... " " 16th "
" " 16th " 1,346,269
2,178,309.... " " 17th "
" " 17th " 3,524,578
5,702.887.... " " 18th "
" " 18th " 9.227,465
14,930,352.... " " 19th "
" " 19th " 24.157,817
39,088,169.... " " 20th "
" " 20th " ......63,245,986. *Answer.*

25. The roof of a building with perpendicular front makes with the horizon an angle of 45°. A leaden ball rolled from the apex thereof strikes the horizontal plane below 40 feet from the base of the front; but when rolled from the center of the roof it strikes only 30 feet from the base. Required the height of the front and the length of the roof. (Porter's Arithmetic.)

Solution by Professor E. B. Seitz. Let AB represent the roof, BD the front, C the center of AB, E the point at which the ball strikes the horizontal plane when rolled from A, and F the point at which it strikes when rolled from C. Draw EG and FH perpendicular to DE, and meeting AB produced in G and H; and draw GKL and HI perpendicular to BD.

Because the angle $BGL = 45°$, $BL = LG = DE = 40$ feet, $BI = HI = DF = 30$ feet, $HK = GK = EF = 10$ feet, $BG = 40\sqrt{2}$, $BH = 30\sqrt{2}$. The velocity attained by a body falling freely or down an inclined plane varies as the square root of the distance described; hence the velocity acquired in rolling down AB : the velocity acquired in rolling down CB :: $\sqrt{2}$: 1. But with the velocities acquired in rolling down AB and CB the ball would describe, respectively, the distances BG and BH, and in the same times that it would fall through the distances GE and HF under the influence of gravity; hence, since $BH = \frac{3}{4} BG$, and the velocities are to each other as $\sqrt{2}$ and 1, the time of describing BH is equal to $\frac{3}{4}\sqrt{2}$ times the time of describing BG, or the time of falling through HF is equal to $\frac{3}{4}\sqrt{2}$ times the time of falling through GE.

Assume 1 second to be the time of falling through GE; then we would have $GE = 16\frac{1}{12}$ feet $= \frac{1}{2}g$, $HF = \frac{1}{2}g \times (\frac{3}{4}\sqrt{2})^2 = \frac{9}{16}g$, and $HK = HF - GE = \frac{1}{16}g$. But $HK =$ 10 feet; hence the distance a falling body describes varies

as the square of the time; the time of falling through $GE = \sqrt{10 \div \frac{1}{16}g} = 4(\sqrt{10 \div g})$ sec.; $\therefore GE = \frac{1}{2}g \times 160 \div g =$ 80 feet, and $BD = BL + GE = 120$ feet.

The velocity acquired in rolling down $AB = BG \div 4\sqrt{10 \div g} = 2\sqrt{5g}$; hence, since the velocity acquired in 1 second is $\frac{1}{2}g\sqrt{2}$, we have $AB = (2\sqrt{(2g)})^2 \div g\sqrt{2} = 10\sqrt{2} = 14.142$ feet.

26. A and B start at opposite points to skate to the other's starting point; distance 8 miles. A, by having the advantage of a uniform wind, performs his task $2\frac{1}{2}$ times the quickest and 48 minutes the soonest. Required the force of the wind per minute, and the time that each is skating it. (Parke's Philosophy of Arithmetic.)

Solution. Their speeds are as 5 : 2, and their difference $3 = 48$ minutes, once $= 16$. Therefore A skates the distance in $16 \times 2 = 32$ minutes, and B in $16 \times 5 = 80$ minutes; consequently A's speed $= 8 \div 32 = \frac{1}{4}$ of a mile a minute, and B's speed $= 8 \div 80 = \frac{1}{10}$ of a mile a minute; and the velocity of the wind $= \frac{1}{4} - \frac{1}{10} = \frac{3}{40}$ of a mile a minute.

27. A man bought a farm for \$6000, and agreed to pay principal and interest in three equal annual installments. What was the annual payment, interest being 6%?

Solution. \$6000 at 6% compound interest in three years will amount to \$7146.096; then we have \$1.00 + \$1.06 + \$1.1236 = \$3.1836; whence, \$7146.096 ÷ \$3.1836 = \$2244.658 +

28. If 12 oxen eat up $3\frac{1}{3}$ acres of pasture in 4 weeks, and 21 oxen eat up 10 acres of like pasture in 9 weeks: to find how many oxen will eat up 24 acres in 18 weeks.

Remark. This problem in its generalized form originated with Sir Isaac Newton. It has appeared in many arithmetics and algebras since.

The celebrated mathematician, Dr. Artemas Martin, Washington, D.C., gives the following beautiful analytical solution: In the first case an ox eats $\frac{1}{4}$ of $\frac{3\frac{1}{3}}{12} = \frac{5}{72}$ of an acre, and $\frac{5}{18}$ of the growth of that acre, in one week; in the second case one ox eats $\frac{1}{9}$ of $\frac{10}{21} = \frac{10}{189}$ of an acre, and $\frac{10}{21}$ of what grows on one acre, in one week.

Since one ox eats the same quantity of grass in one week in each case, therefore $\frac{10}{21} - \frac{5}{18} = \frac{25}{126}$ of the growth of one acre during one week is $= \frac{5}{72} - \frac{10}{189} = \frac{25}{1512}$ of an acre; and $\frac{25}{1215} \div \frac{25}{126} = \frac{1}{12}$ of an acre, what grows on an acre during one week.

$\frac{5}{72} + \frac{5}{18}$ of $\frac{1}{12} = \frac{5}{54} =$ of an acre, the part of the original quantity on one acre which one ox eats in one week.

$\frac{5}{54} \times 18 = \frac{5}{3} =$ quantity of grass, in acres, one ox will eat in 18 weeks.

$24 + (\frac{1}{12} \times 24 \times 18) = 60 =$ quantity of grass, in acres, to be eaten from 24 acres in 18 weeks; and $60 \div \frac{5}{3} = 36$, the number of oxen required to eat it.

The following is an easy method of solving all such problems:

1. Suppose each ox to eat 100 pounds of grass in 1 week; then 12 oxen will eat 4800 pounds in 4 weeks; and $4800 \div 3\frac{1}{3} = 1440$ pounds, the whole quantity, including the growth on 1 acre for 4 weeks.

2. 21 oxen will eat, at the same rate, 18,900 pounds in 9 weeks, and $18{,}900 \div 10 = 1890$ pounds, the whole quantity, including the growth on 1 acre for 9 weeks. Since there was the same quantity of grass on each acre at the time the oxen began to graze, then $1890 - 1440 = 450$ pounds must be the growth on 1 acre for 5 weeks, and $450 \div 5 = 90$ pounds is the growth on 1 acre in 1 week; and the original quantity of grass on 1 acre $= 1440 - 360 = 1080$ pounds. Consequently, $1080 \times 24 = 25{,}920$ pounds, the original quantity on 24 acres, and $90 \times 18 \times 24 = 38{,}880$ pounds,

the growth on 24 acres, and for 18 weeks $25{,}920 + 38{,}880 = 64{,}800$, the total number of pounds on 24 acres in 18 weeks.

3. Since an ox eats 100 pounds in one week, in 18 weeks he will eat 1800 pounds; therefore, $64{,}800 \div 1800 = 36$ oxen, the number required.

Remark. Any other number may be assumed as the number of pounds an ox eats in one week.

Algebra.

Brief History.

The word Algebra is of Arabic origin, and in its original form it is Algabr, which by a slight change becomes Algebra. Literally it means the reduction of parts to the whole; but for present purposes it may be defined as that branch of mathematics which treats of the general relation of quantities by means of symbols.

The history of this science is interesting and instructive, and like all other sciences it has been developed gradually from a few elementary notions. Not far from the year 350 A.D., Diophantus, a Greek, wrote a commentary on Arithmetic. In this work, a mutilated copy of which was discovered about the middle of the sixteenth century at Rome in the Vatican library, are solved some equations of the first and second degrees. He simply wrote out his solutions, not using any of the signs now employed.

The problems that Diophantus solved, belonging to our elementary Algebra, were of the following forms:

$$(1)\quad x+y=a,$$

and $$(2)\quad x^2+y^2=b, \text{ to find } x \text{ and } y;$$

also,

$$(1)\quad x-y=a,$$

and $$(2)\quad x^2+y^2=b, \text{ to find } x \text{ and } y.$$

These equations lead to the simplest forms of quadratics; but if the solutions were written out in full without the aid of symbols, the processes would be quite tedious; hence it is not a matter of great surprise that no more progress

was made. But in another direction Diophantus achieved greater success, and in fact he laid the foundation for that interesting class of problems now called "The Indeterminate Analysis," which relates chiefly to square, cube, and biquadrate numbers, and to rational right-angled triangles and other polygonal figures.

The fragments of Algebra that have come to us from Diophantus show that he was a mathematician of no ordinary skill for that age. His solutions of indeterminate problems of the second degree effected by the cumbersome methods he employed attest his ability, and the fact that the science remained stationary, say, for a thousand years, as it came from his hands, is the strongest evidence of his analytical power.

As an illustration of what Diophantus was enabled to accomplish, I select a problem which he solved, namely:

"To divide a given square number, a^2, into two other squares."

Solution. Let x^2 be one of the required squares, and $a^2 - x^2$ the other.

Put $a^2 - x^2 = (a - bx)^2$, and $x = \dfrac{2ab}{b^2+1}$.

Substituting the value of x, we have

$$x^2 = \frac{4a^2b^2}{(b^2+1)^2}, \quad \text{and} \quad a^2 - x^2 = \frac{a^2(b^2-1)^2}{(b^2+1)^2}.$$

The values of a and b may be assumed at pleasure. If $a = 4$, $b = 3$, $x = \frac{12}{5}$; also, if $a = 10$, $b = 2$, $x = 64$. This solution is in the ordinary algebraic form.

Prof. G. Gill originated a new method of solving such problems. Here is his solution. Given $x^2 + y^2 = a^2$, so that if we take $x = a \sin A$, $y = a \cos A$; we shall have

$$x^2 + y^2 = a^2 (\sin^2 A + \cos^2 A) = a^2.$$

A is any angle whose functions are rational numbers.

Dr. Artemas Martin solved the same problem as follows:

Solution. Let x^2 and y^2 be the numbers; then

$$x^2 + y^2 = \square = z^2. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation is satisfied by $x = z \sin \phi$, $y = z \cos \phi$. Take, therefore,

$$\cot \tfrac{1}{2} \phi = \frac{m}{n} \quad \text{and} \quad z = m^2 + n^2;$$

then

$$x = 2mn \quad \text{and} \quad y = m^2 - n^2.$$

If $m = 2$, $n = 1$,

$$x = 4, \quad y = 3, \quad z = 5,$$

which are the least integral numbers.

The expressions $m^2 + n^2$, $m^2 - n^2$, and $2mn$ represent the three sides of a right-angled triangle, and no doubt it was owing to this discovery that the problem, "to find the sides of a right-angled triangle in integral numbers," suggested itself to mathematicians.

Diophantus placed in his treatise the law of the *minus sign* when he announced that "minus multiplied by *minus* produces *plus*," the truth of which is still accepted by most authors of the present without an attempt at an explanation satisfactory to the beginner.

Hypatia, the celebrated daughter of Theon, wrote a com mentary on the work composed by Diophantus, but this was lost, as well as a similar treatise that she had prepared on conic sections. Her knowledge of philosophy and mathematics caused both men and women to become jealous of her, and she was torn to pieces with a harrow,—some assert, however, with hot pincers.

About this time Rome broke into two great empires. In the dissolution of a thousand years, algebra was not cultivated except by the Arabs. What the people in India were doing, we know not. In the library of Oxford is a manu-

script copy of an Arab Algebra bearing the date 1342. This is the oldest copy of an Arab Algebra there is in Europe. The evidence appears quite conclusive that the Arabs derived their knowledge of this branch from the Hindus. Yet recent research tends to establish the fact that the Hindus made little progress in the science compared to the advancement made by the modern nations of Europe and of this country.

The first one to introduce Algebra, or rather to revive it, in Europe was Leonardo, a merchant of Pisa. He had traveled extensively in the East, where he picked up some information on the subject of Algebra, and upon his return he taught pupils, and afterwards published two treatises which were written in Latin verse. His knowledge of the subject was confined to the treatment of a few special equations of the first degree.

It was not till 1505 that an equation of the third degree was solved. Ferreus, a professor of mathematics, Bologna, was the first to solve this problem in special cases. In those days it was the custom when one had made a new discovery to conceal it from others, and then to frame an arithmetical problem involving the discovery, and send it as a challenge. Ferreus kept his secret nearly thirty years, when he communicated it to a pupil of his, a Venetian, Florido, who sent it as a challenge to Tartalea of Brescia. Tartalea had not only solved the problem proposed, but he had solved three other special cases. He in turn challenged Florido, who failed.

Contemporary with Tartalea was Cardan, a man of great ingenuity and skill. Under an oath of secrecy he obtained Tartalea's method of solving cubic equations, and then published it as his own. All algebraists are familiar with Cardan's (Tartalea's) formula for the resolution of cubic equations.

At this time it does not appear that a general solution

of a complete equation of the second degree had been effected. Every complete equation of this degree may be placed under one of the four forms:

$$\begin{aligned}
&(1)\quad x^2 + 2px = q;\\
&(2)\quad x^2 + 2px = -q;\\
&(3)\quad x^2 - 2px = q;\\
&(4)\quad x^2 - 2px = -q.
\end{aligned}$$

These four forms are comprehended under the more general equation

$$x^2 \pm 2px = \pm q,$$

and whose roots are:

$$\begin{aligned}
\text{For } &(1)\quad x = -p \pm \sqrt{q + p^2};\\
&(2)\quad x = p \pm \sqrt{q + p^2};\\
&(3)\quad x = -p \pm \sqrt{-q + p^2};\\
&(4)\quad x = p \pm \sqrt{-q + p^2}.
\end{aligned}$$

Forms (1), (2), (3), and (4) are included under the more comprehensive equation $x^2 \pm 2px = \pm q$, which had not yet been solved by the keenest analysts when Scipio Ferreus solved the special cubic equation $x^3 + bx = c$. He had found how to make the second term of a cubic equation disappear by substitution, and thus to reduce certain equations to the special form above. His solution was as follows:

Assume $y + z = x$, and $3yx = -b$. Then substituting these values in $x^3 + bx = c$, it becomes

$$\begin{aligned}
&y^3 + 3y^2z + 3yz^2 + z^3 + b(y + z) = c =\\
&y^3 + z^3 - b(y + z) + b(y + z) = y^3 + z^3 = c.
\end{aligned}$$

Squaring $y^3 + z^3 = c$, and subtracting four times the cube of $yx = -\frac{b}{3}$ from it, we have

$$y^6 - 2y^3z^3 + z^6 = c^2 + \tfrac{4}{27}b^3,$$

or

$$y^3 - z^3 = \sqrt{c^2 + \frac{4b^3}{27}}.$$

By addition,

$$y^3 = \frac{c}{2} + \sqrt{c^2 + \frac{4b^3}{27}},$$

and

$$y = \sqrt[3]{\frac{c^2}{2}\sqrt{c^2 + \frac{4b^3}{27}}};$$

$$z^3 = \frac{c^2}{2} - \sqrt{c^2 + \frac{4b^3}{27}},$$

and

$$z = \sqrt[3]{\frac{c^2}{2} - \sqrt{c^2 + \frac{4b^3}{27}}};$$

whence

$$y + z = x = \sqrt[3]{\frac{c^2}{2} + \sqrt{c^2 + \frac{4b^3}{27}}} + \sqrt{\frac{c^2}{2} - \sqrt{c^2 + \frac{4b^3}{27}}},$$

which gives one root of the equation.

Tartalea solved the equations $x^3 - bx = c$, and $x^3 - bx = -c$, as well as the equation $x^3 + bx = c$; but Cardan received, undeservedly, the credit. Cardan's formula fails when $c^2 + \frac{4b^3}{27}$ is negative, and yet the three roots are real and unequal, as may be easily shown. But that particular form of the cubic equation in which the roots are all *real* is called the "Irreducible Case," while the "Reducible Case" is encountered when two of the roots are imaginary or are equal. Having obtained one value of the unknown quantity in a cubic equation, the cubic can then be depressed to one of the second degree, and solved as a quadratic.

Every cubic must have at least one real root; imaginary roots enter by pairs.

Biquadratic Equations.

Mathematicians next turned their attention to equations of the fourth degree. The newly discovered methods for solving cubics were not applicable to biquadratics. Some contended that a general solution of a complete equation of the fourth degree was impossible. Cardan did not think so, and he asked his pupil, Lewis Ferrari, a young man of remarkable analytic skill, to discover a solution. Ferrari not only solved the special problem which others were unable to reduce, but he likewise effected a solution of the general problem. He made the solution of the equation of the fourth degree depend upon the solution of a cubic.

His method of reduction was the following:

Let $x^4 + ax^3 + bx^2 + cx + d = 0$ be a complete biquadratic equation. Assume

$$(x^2 + \tfrac{1}{2}ax + p)^2 - (qx + r)^2$$
$$= x^4 + ax^3 + bx^2 + cx + d. \quad . \quad . \quad (1)$$

Expanding, we have

$$x^4 + ax^3 + (2p + \tfrac{1}{4}a^2 - q^2)x^2 + (ap - 2qr)x + p^2 - r^2$$
$$= x^4 + ax^3 + bx^2 + cx + d. \quad . \quad . \quad (2)$$

Equating like coefficients of the same powers of x, we have

$$2p + \frac{a^2}{4} - q^2 = b \ldots (3) = 2p + \frac{a^2}{4} - b = q^2;$$
$$ap - 2qr = c \ldots (4) = ap - c \qquad = 2qr;$$
$$p^2 - r^2 = d \ldots (5) = p^2 - d \qquad = r^2.$$

Since the product of the absolute terms of (3) and (5) is equal to $\frac{1}{4}$ of the square of the absolute term of (4), we have

$$2p^3 + \left(\frac{a^2}{4} - b\right)p^2 - 2dp - d\left(\frac{a^2}{4} - b\right)$$

$$= \tfrac{1}{4}(a^2p^2 - 2acp + c^2), \quad . \quad . \quad . \quad (6)$$

or

$$p^3 - \frac{bp^2}{2} + \tfrac{1}{4}(ac - 4d)p = \tfrac{1}{8}(c^2 + a^2d) - \frac{bd}{2}. \quad . \quad (7)$$

p can be found under the solution of the cubic as heretofore explained.

Also, from (3),

$$q = \sqrt{2p + \frac{a^2}{4} - b},$$

and from (4) and (5),

$$r = \frac{ap - c}{2q} = \sqrt{p^2 - d}.$$

Again,

$$x^4 + ax^3 + bx^2 + cx + d = (x^2 + \tfrac{1}{2}ax + p)^2 - (qx + r)^2 = 0.$$

It is evident that

$$(x^2 + \tfrac{1}{2}ax + p)^2 = (qx + r)^2, \quad . \quad . \quad . \quad (8)$$

and

$$x^2 + \tfrac{1}{2}ax + p = qx + r,$$

or

$$x^2 + (\tfrac{1}{2}a - q)x = r - p.$$

Substituting the above values of p, q, and r and arranging, we have

$$x^2 + \left(\tfrac{1}{2}a \mp \sqrt{2p + \frac{a^2}{4} - b}\right)x + p \mp \sqrt{p^2 - d} = 0$$

when $ap - c$ is positive, and

$$x^2 + \left(\tfrac{1}{2}a \mp \sqrt{2p + \frac{a^2}{4} - b}\right)x + p \pm \sqrt{p^2 - d}$$

when $ap - c$ is negative. These two quadratics give the four roots of the proposed equation.

If we put $p = z + \frac{b}{6}$, and substitute in (7), it becomes, after reduction,

$$z^3 + \left(\frac{ac}{4} - \frac{b^2}{12} - d\right)z = \frac{b^3}{108} + \tfrac{1}{8}(c^2 + da^2) - \frac{b}{24}(ac + 8d). \quad (8)$$

One root of this cubic can now be found, and the equation depressed afterwards to a quadratic, when the remaining values can be ascertained. Since the equation of the fourth degree is made to depend upon the solution of a cubic, the same difficulties are encountered in regard to the "Irreducible Case" as in the original cubic.

As an illustration of this method, let there be given

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0,$$

to find all its roots.

$$a = -10, \quad b = 35, \quad c = -50, \quad d = 24.$$

Substituting these values in the general form of (8) and reducing, it becomes

$$z^3 - \tfrac{13}{12}z = \tfrac{3}{108},$$

which solved by the rule previously given for binomial surds, $z = \frac{7}{6}$. Solving the resulting quadratics, the roots 1, 2, 3, 4 are obtained.

Notwithstanding the resolution of the biquadratic equation, the true nature of equations was not yet understood. Cardan himself was unable to explain the "Irreducible Case," which was not explained till Bombelli, an Italian mathematician, published his treatise on Algebra in 1572, and he noted the fact that the algebraic solution of this "case" corresponded to the ancient problem, "how to trisect any angle."

Bombelli also solved the expression how to extract the cube root of $a + \sqrt{-b}$.

The following rule, which bears his name, states his process:

"First find $\sqrt{a^2 + b}$; then, by trials, search out a number, c, and a square, $\sqrt{d}$, such that the sum of their squares, $c^2 + d$, be $= \sqrt[3]{a^2 + b}$, and also $c^3 - 3cd$ be $= a$: then shall $c + \sqrt{-d}$ be the cube root of $a + \sqrt{-b}$ sought."

This rule will be applied in the solution of the following problem, namely: Extract the cube root of $9 + 25\sqrt{-2}$. Here $a = 9$, $b = 1250$; hence $\sqrt[3]{a^2 + b} = \sqrt{81 + 1250} = 11$; also, $c^2 + d = 11$, and $c^3 - 3cd = 9$, which are true when $c = 3$, $d = 2$; therefore the cube root is $3 + \sqrt{-2}$. *Ans.*

Contemporary with Cardan and Tartalea were two German mathematicians, Stifelius and Schenbelius, who, independent of what the Italians had done, did much to improve algebraic notation. The symbols for addition, subtraction, and the square root were first employed by Stifelius. His chief work was published in 1544.

Robert Recorde, a teacher of mathematics and doctor of medicine, published the first algebra in the English language, which he called the "Whetstone of Wit," etc. He introduced the sign of equality.

Vieta, a celebrated French mathematician born 1540, made algebra a symbolical science, and he was the first to represent known quantities by general characters or letters. This notation at once made a special solution-general, and saved the trouble of going over each problem as it was proposed. One solution, when general characters are employed, solves all problems of that class. The result is embraced in a general formula, and the numerical values substituted in the formula will satisfy the conditions whatever the nature of the special problem. Vieta's method included geometrical problems as well as algebraic, and thus led to great improvements in other departments of

mathematics. He was the first to solve equations by approximation. Vieta died at the age of sixty-three. He printed his writings at his own expense, and liberally bestowed them on men of science.

Prior to Vieta's time, Albert Girard, a Flemish mathematician, had made considerable progress in algebra beyond what others had done. He was the first to interpret the negative sign in geometrical problems, and to point out the nature of imaginary quantities, and to call attention to the fact that there are as many roots in an equation as there are units in its degree.

Thomas Harriott, born at Oxford 1560, was an eminent mathematician in his day, and he did much toward furthering the science of algebra. His most important contribution was that every equation was compounded of as many equations as there are units expressing its order. For instance, a quadratic is composed of the product of two simple equations, a cubic of three, and so on. Harriott gave compactness to algebraic language.

By slow steps the science of algebra was developed. Each discovery required time for its exploration and its application. Rarely did it happen that the one who made a discovery lived to complete it. Vieta applied algebra to geometry, but it was left to Descartes to apply and to interpret the doctrine of curves. He expressed the relation of lines by the aid of algebraic symbols, which constitute the equation of the curve. Hallam in speaking of Descartes says: "One man, the pride of France and wonder of his contemporaries, was destined to flash light upon the labors of the analyst and point out what those symbols, so darkly and painfully traced, might represent and explain."

Descartes observed also that, in any complete equation, when the roots are all real the number of positive roots is equal to the number of variations of signs, and the num-

ber of negative roots is equal to the number of permanences of signs. This is known as "Descartes' Rule."

Newton gave to the world the "Binomial Theorem," but left no rigorous demonstration of it, and at the beginning of the present century Lagrange, one of the most distinguished of the French mathematicians and promoters of algebra, discovered that every numerical equation has a root, either real or imaginary, which, when substituted for the unknown quantity, will reduce the equation to zero. Gauss, the German analyst, in 1801 also developed the subject of "Binomial Equations," and in 1819 William G. Horner, Bath, England, published his method of solving numerical equations of any degree.

Following closely upon the discovery by Horner, Sturm developed the beautiful theorem which bears his name, the object of which is to find the limits and situation of the real roots of an equation. Of late years so many discoveries have been brought to light in this science by able analysts that the reader is referred to the recent publications of Europe and of this country for further information. So great have been the extensions of this science that the modern higher algebra may be regarded as the highest achievement of mathematical skill.

On Teaching Algebra.

For a pupil to drop arithmetic and to commence algebra is a leap from the known into the unknown. A new language and strange and unmeaning terms confront him upon every hand. Little of what he learned in arithmetic is related to anything he finds in this new field of investigation. Some terms, it is true, bear resemblances to certain features of arithmetic, yet in such a remote manner that the connection cannot be traced. Heretofore he worked with numbers which he added, subtracted, multi-

plied, divided, raised to powers, and extracted their roots; now he deals with letters and signs in a dim sort of way, with but little light to help him. Unsteady are his steps at the beginning; and so slow is the process by which the mind, accustomed by long practice to run in fixed lines, turns from old modes of thought to the new ones, that he spends much of his time in trying to adjust himself to his strange surroundings. The chief difficulty for him is to think in algebraic language by the extension of arithmetical nomenclature; but if the teacher himself is skillful in closely connecting the two sciences and in showing how the processes of the one are related to the processes of the other and helps to explain them, the pupil's progress is easy and rapid. The unrelated presentation of algebra is the chief cause of perplexity and discouragement to the pupil.

There are two distinct methods, or plans, teachers pursue in leading their pupils in an attack on algebra. One class of teachers manifest great haste in putting their pupils to work and devote little time to definitions and principles. Pupils under this system of tuition frequently work through an elementary algebra without being able to give one correct definition, and their knowledge of what they have gone over is vague and unsatisfactory.

The other method assumes that when a pupil is properly prepared to begin algebra, he has reached such a degree of mental development as will enable him to master the subject easily. Maturity of mind is required, and precise and clear-cut thinking is demanded. Algebra is not to be studied in order to throw light on the shadier portions of arithmetic, but as an enlarged or unabridged edition of arithmetic.

As a science, then, arithmetic should have been thoroughly mastered, and especially mental arithmetic, before the pupil is permitted to commence algebra.

In beginning algebra, the preliminary definitions should first occupy the pupil's attention. Definitions are therefore to be learned, explained, and understood. Definitions are "hitching-posts," and the learner or teacher who has his definitions ready to hand is doubly equipped for attack or defense. Besides, there is a strong educational value, as well as a logical value, attaching to exact definitions. Neither is it supposed that in learning a definition its full import is apprehended by the learner at once; but rather as his mental horizon of the subject enlarges, he refers backward to the definition and he sees that it still holds true. Were the mind so constituted that by learning definitions the whole subject would instantly unfold itself, then there would be nothing for the mind to do except to learn the definitions and not the subject. Definitions are the broadest generalizations of the human mind, and the strength, the beauty, the elegance, and the universality of a good definition depend upon its perfect adaptability to every stage of the learner's knowledge of the subject. And here I will remark that correct mathematical definitions are less variable than any other class of definitions in other departments of learning.

In teaching algebra it is maintained that the true test of successful teaching is measured by the knowledge of the entire class, rather than by the achievements of the very few, or perhaps of one member only.

In the remainder of this paper I will emphasize the topics which appear to me as the most important for the pupil to know in order to insure his progress.

What does a definition, an explanation, a solution, or a principle mean to a pupil? How does the matter, whatever it may be, appear to him, and how does he understand it? It is the pupil's conception of the subject that must be sought for, not the teacher's.

The following should be understandingly learned by the pupil:

1. Algebra is that branch of mathematics in which the relations of quantities are expressed and investigated, and the reasoning abridged and generalized by means of figures, letters, and signs.

2. Figures, letters, and signs are called symbols.

3. Symbols are divided into three classes: 1. Symbols of Quantity; 2. Symbols of Operation; 3. Symbols of Relation.

4. Symbols of Quantity are figures and letters. Figures and the first letters of our alphabet denote *known* quantities; and *unknown* quantities are denoted by the final letters of our alphabet. Sometimes Greek letters are employed to denote either known or unknown quantities.

Zero (0) denotes the absence of quantity, while ∞ is the symbol called infinity, or a quantity without limit. Accents and subscripts are attached to symmetrical quantities; thus, a', a'', a''', and a_1, a_2, a_3, etc.

5. The Symbols of Operation are $+$, $-$, $\times$ or [.], $\div$, $\lfloor$, ——,), :, a^2, a^3, ... x^n, $\sqrt{}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... $_n$.

6. The Symbols of Relation are $=$, $>$, $<$, :, : :: :, $\propto$, |, ——, ( ), [ ], { }, ..., $\therefore$, $\because$

The meaning of the symbols must be thoroughly mastered at the outset. They are the mathematical alphabet, and progress is impossible without them.

7. The pupils must learn the meaning and the application of the following: 1. Coefficient; 2. Exponent; 3. Power; 4. Root; 5. Reciprocal; 6. Positive Term; 7. Negative Term; 8. Simple Term; 9. Compound Term; 10. Similar Terms; 11. Dissimilar Terms; 12. Homogeneous Terms; 13. Monomial; 14. Binomial Trinomial; 15. Polynomial; 16. The Degree of a Term.

The technique of the science should be mastered at once. Exact knowledge at the beginning is the true motto.

Suggestions.

1. What is the difference between the coefficient and the exponent of $3x^3$? Which is the exponent? Which the coefficient?

2. Compare $7a^2bx^2y^{\frac{1}{2}}$ and $-9ab^2x^{\frac{1}{2}}y^2$, and also compare each with $6a^2bcx^{\frac{1}{2}}y^{\frac{1}{2}} + 9b^2c^2x^{\frac{2}{3}}y^{\frac{3}{2}}$ by indicating their agreements and differences as referred to the preceding definitions. Let no confusion of thought linger in regard to the difference between "coefficient" and "exponent," "similar" and "dissimilar" terms, and "positive" and "negative" signs.

Teach everything for all time.

8. In teaching Addition, Subtraction, Multiplication, and Division, let the pupil compare each process with the corresponding process in arithmetic, thus:

1. Add ax, bx, $-cx$, dx.

2. Add 21, 15, $-8 + 6$.

What are the agreements in the two processes? What are the differences? Does the algebraic process include anything not contained in the arithmetical process? What is it?

3. In what respects do Addition and Subtraction in algebra differ? Why are they not the same? What is the difference in regard to the use of the positive and negative signs in Addition and in Subtraction? How are like terms added or subtracted? How are their coefficients affected? How their exponents? Give the law of the signs in Addition; in Subtraction.

4. Removal or Introduction of Parentheses. 1. When the expression is preceded by *plus;* 2. When the expression is preceded by *minus;* 3. When expressions occur with more than one parenthesis, or parentheses within a parenthesis; 4. To put any number of terms within a

parenthesis; 5. To put any number of terms within a parenthesis, and the *minus* sign placed before it.

5. To translate algebraic expressions into ordinary language, and to translate ordinary language into algebraic expressions, thus:

1. Find the sum of

$$3(a^2 - by)^{\frac{1}{2}} - 2\sqrt{a(a - by)} + 7(a-by)^{\frac{1}{2}},$$

and write the result in ordinary language.

2. If a equals 3, b equals 2, y equals $2\frac{1}{2}$, what is the numerical value of the expression?

3. If y equals 0, and the values of a and b are unchanged, what is the value?

4. If a equals $\frac{1}{2}$, b equals $\frac{1}{3}$, y equals $\frac{1}{4}$, find the value.

9. Multiplication and Division. I. *In Multiplication* the pupil must look out for the following points: 1. The coefficients, numerical and literal, and their signs, as they occur in the partial or in the total product; 2. The arrangement of the literal factors; 3. The exponents of the quantities; 4. Contrast the law of the exponents when like literal factors are employed in Multiplication with the law in Addition and in Subtraction; 5. Show the difference in Division in regard to the exponent in the quotient as compared with the exponent in the product in Multiplication. 6. What does Division include that is not found in Multiplication? 7. Is there any similarity in pointing decimals in Arithmetic and the laws of the exponents in the four fundamental rules of Algebra? 8. How do they agree? 9. How do they differ?

II. *The Laws of the Signs.* 1. Like sign multiplied by like sign; 2. Sign multiplied by unlike sign; 3. Like sign divided by like sign; 4. Sign divided by unlike sign. 5. The Generalized Laws are: (1) Like signs give plus; (2) Unlike signs give minus; (3) If the number of minus factors of a product be odd, the product is odd; (4) An odd

power of a negative quantity is minus; (5) The effect of changing the sign of a factor, or of changing the sign of the dividend or of the divisor.

The pupil should make haste slowly, but surely, in Addition, Subtraction, Multiplication, and Division. Everything ought to be thoroughly understood by him except the full force of the definition of Algebra, which he only partially apprehends now, before he takes up additional topics. Frequent reviews, references, cross-references, comparisons, and entire familiarity with algebraic notation, are absolutely required. The subject must be cleared of all doubts and difficulties. There must be no uncertainty in the pupil's mind. Each principle, law, operation, deduction, must be firmly grasped and held in the mind, ready for instant use.

Problems graduated according to complexity and difficulty help to develop original power. One problem changed as to some one of its conditions each time will give a better view of a subject than many different problems. One new step at a time is best for the average mind. Progress is measured by the pupil's skill in mastering original problems. Judicious practice that stimulates but does not discourage is the sure road to success.

Theorems.

10. The theorems are employed chiefly to contract work. It is more convenient to work with small expressions than with complicated ones. Certain forms occur so frequently that the pupil should learn them once for all. The chief elementary theorems should be known and remembered. Their importance cannot be overestimated.

THEOREM 1. The square of the sum of two quantities is equal to the sum of their squares plus twice their product. Thus,

$$(20+5)\,(20+5)=400+25+200=625\,;$$

or

$$(a+b)\,(a+b) = (a+b)^2 = a^2 + b^2 + 2ab = a^2 + 2ab + b^2.$$

THEOREM 2. The square of the difference of two quantities is equal to the sum of their squares minus twice their product. Thus,

$$(20-5)\,(20-5) = 400 + 25 - 200 = 225\,;$$

or

$$(a - b\,(a-b) = (a-b)^2 = a^2 + b^2 - 2ab = a^2 - 2ab + b^2.$$

Theorems 1 and 2 are symbolically expressed thus:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

THEOREM 3. The product of the sum and difference of two quantities is equal to the difference of their squares. Thus,

$$(20+5)\,(20-5) = 400 + 100 - 100 - 25 = 400 - 25;$$

or

$$(a+b)\,(a-b) = a^2 + ab - ab - b^2 = a^2 - b^2.$$

THEOREM 4. The difference of the same powers of two quantities is divisible by the difference of the quantities. Thus,

$$(a^2 - b^2) \div (a-b) = a + b.$$
$$(a^3 - b^3) \div (a-b) = a^2 + ab + b^2.$$
$$(a^4 - b^4) \div (a-b) = a^3 + a^2b + ab^2 + b^3 = (a^2 + b^2)\,(a+b).$$
$$(a^5 - b^5) \div (a-b) = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

The pupil will observe the law by which the exponent of the leading letter decreases, while the exponent of the other increases; also, that the sum of the exponents in the same expansion is invariable, and that the number of terms in each expansion is equal to the number of units in the exponent.

THEOREM 5. The difference of the same even powers of two quantities is divisible by the sum of the quantities. Thus,

$$(a^2 - b^2) \div (a + b) = a - b.$$
$$(a^4 - b^4) \div (a + b) = a^3 - a^2b + ab^2 - b^3.$$
$$(a^6 - b^6) \div (a + b) = a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5.$$

Theorems 4 and 5 may be extended when the exponents in the dividend are different, if the same difference exists in the divisor. Thus,

$$(a^9 - b^6) \div (a^3 - b^2) = a^6 + a^3b^2 + b^4.$$
$$(a^8 - b^{12}) \div (a^2 + b^3) = a^6 - a^4b^3 + a^2b^6 - b^9.$$

The law in regard to the exponents can be extended to coefficients, if the coefficients of the dividend are the same as the corresponding terms of the divisor. Thus,

$$(256a^{12} - 81b^8) \div (4a^3 + 3b^2) = 64a^9 - 48a^6b^2 + 36a^3b^4 - 27b^6.$$

THEOREM 6. The sum of the same odd powers of two quantities is divisible by the sum of the quantities. Thus,

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2.$$
$$(a^5 + b^5) \div (a + b) = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

Also,

$$(a^{12} + b^9) \div (a^4 + b^3) = a^8 - a^4b^3 + b^6.$$

In the case of coefficients the extension is the same as in Theorem 5.

Let the pupil compare Theorems 4, 5, and 6. Drill upon these theorems till they are well understood.

THEOREM 7. The product of two binomials whose first terms are the same is equal to the square of the first term, plus the sum of the second terms into the first, plus the algebraic product of the second terms.

Thus:

$$x^2 + 11x + 30 = (x + 5)(x + 6).$$

Here x is the first term of both binomials, the sum of the second terms is 11, and their product is 30. This theorem lays the foundation for the solution of all numerical equations above the first degree.

Again, suppose we have $x^2 - 8x - 20$ to be factored. We must find two numbers whose sum is -8 and whose product is -20. Evidently

$$-10 + 2 = -8, \quad \text{and} \quad -10 \times 2 = -20,$$

whence

$$x^2 - 8x - 20 = (x + 2)(x - 10);$$

or regarding the expression as a quadratic equation, we have

$$x^2 - 8x - 20 = (x + 2)(x - 10) = 0;$$

whence each factor equal zero, or

$$x = -2, \quad \text{and} \quad x = 10.$$

Drill the pupil on exercises under this theorem. Select the exercises from the text he studies. Such a direction given to his work will save time frequently in solving equations by other and more formidable methods. Factoring is "the short-cut" in the solution of problems, and skill in handling equations depends almost entirely upon the operator's tact in discovering and in suppressing common factors. In short, it is the ability to discover latent forms and to reject them while preserving the value of the expression.

But it would be wrong for the pupil to conclude that all trinomials are factorable. The forms that can be resolved are:

$$(1)\quad x^2 + ax + b; \qquad (3)\quad x^2 + ax - b;$$

$$(2)\quad x^2 - ax + b; \qquad (4)\quad x^2 - ax - b.$$

In (1) and (2) a is the sum of two quantities whose product is b; and in (3) and (4) a is the difference of two quantities whose product is b. The sum and difference are here used in their arithmetical sense.

Greatest Common Divisor and Least Common Multiple.

11. The Greatest Common Divisor affords a test for ascertaining whether two or more quantities, however complicated, are prime to each other. When it is desirable to suppress a common factor that exists in two or more algebraic expressions, or to search for the existence of such a factor, recourse is had to that process known as finding the Greatest Common Divisor of the expressions. As a process the pupil should know it. All methods designed to abridge or to simplify algebraic expressions ought to be well understood by the pupil. The groundwork of Algebra must be laid in a thorough mastery of the elementary processes of reduction and simplification. These are the "clearing-up processes" for the successful handling of equations.

In one respect the Least Common Multiple is different from the Greatest Common Divisor. It is the least container of two or more quantities, and its use is to reduce dissimilar expressions, chiefly, to similar ones, that they may be more closely united, separated, or partitioned.

The drills in Greatest Common Divisor and Least Common Multiple help to fix the four fundamental operations more clearly in the mind; yet the reasons for learning them will not so readily suggest themselves to the pupil unless constant reference is made to the same operations in Arithmetic; but when the pupil is reminded that dissimilar fractions must be changed to the same unit of value before they can be added, subtracted, or divided,—in short, handled with any degree of satisfaction, the way is partially blazed out for him to trace the connection in the

two branches. Lest the pupil mistake the nature of Factors, Divisors, and Multiples, he is requested to group them under the heading *"Labor-Saving Methods."*

Let the pupil draw a sharp, hard line between how to find the Greatest Common Divisor of two or more quantities and how to find the Least Common Multiple of the same quantities. In teaching, I regard this as better than taking a different exercise for each. The contrast is more striking. Contrasts as well as analogies are helps in setting matters in the mind.

The teacher must not forget to review frequently, and to have the pupil trace every step back to the definitions or principles upon which it depends. This connects the work and gives the pupil confidence in himself, and it cultivates the habit of demanding a reason for everything he does.

Fractions.

Many persons fail to get a clear conception of the full force of the sign of a fraction; but if the use of the "parenthesis" has been taught, little difficulty will be experienced. In looking at an algebraic fraction, the learner should observe four things:

1. The sign that precedes the line between the numerator and the denominator. This sign is called the *apparent* sign of the fraction.

2. The sign of the numerator, or of each term of the numerator.

3. The sign of the denominator, or of each term of the denominator.

4. The sign of the quotient, or of each term of the quotient.

The dividing line answers the purpose of a parenthesis or vinculum, and when the sign that precedes the dividing

line is changed, the sign of the fraction is changed also. Thus:

$$\frac{ab}{a} = b; \quad \text{but} \quad -\frac{ab}{a} = -b;$$

or,

$$\frac{ab - ac}{a} = b - c; \quad \text{but} \quad -\frac{ab - ac}{a} = -b + c.$$

Again,

$$\frac{ab}{a} = b; \quad \text{but} \quad -\frac{-ab}{a} = b;$$

or,

$$\frac{-ab}{-a} = b; \quad \text{or} \quad -\frac{ab}{-a} = b.$$

Two signs when changed do not change the sign of the fraction. But when all three signs are changed, the sign is changed. Thus:

$$\frac{ab}{a} = b; \text{ but } -\frac{-ab}{-a} = -b = -\frac{ab}{a} = \frac{-ab}{a} = \frac{ab}{-a}.$$

Which shows that to change all the signs is the same in effect as changing one sign. That is, to change the sign preceding the dividing line is the same in effect as changing the sign of each term in the numerator and in the denominator of the fraction; and this holds true notwithstanding the number of terms in either the numerator or the denominator. Therefore the pupil should be deeply impressed with this principle, that to change the sign of a fraction from plus to minus, or from minus to plus, changes the sign of each term. Frequently it is very desirable to change algebraic expressions from plus to minus, or from minus to plus, that they may be more easily dealt with.

There is, however, another phase of this subject that is more difficult for the learner to understand than handling terms united by plus or separated by minus, and that is

when the terms of a fraction are composed of any number of factors. Thus:

$$\frac{a-b}{(x-y)(x-z)}=\frac{a-b}{(y-x)(z-x)}=\frac{b-a}{(y-x)(x-z)}=\frac{b-a}{(x-y)(z-x)}.$$

But it will be observed that in each case the signs of two factors have been changed. If the signs of all be changed thus,

$$\frac{b-a}{(y-x)(z-x)},$$

the value of the fraction is changed. Hence the law is, when a fraction is composed of any number of factors, that any even number of factors may have their signs changed without changing the value of the fraction ; but if any odd number of factors have their signs changed, the value of the fraction is changed.

To illustrate this law the following is selected :

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}=?$$

Solution. On account of symmetry it is preferable to make the common denominator $abc(a-b)(a-c)(b-c)$.

Hence we must change the sign of $b-a$ in the second term, and also the signs of $c-a$ and of $c-b$ in the third term. But we cannot change the sign of $b-a$ in the second term unless the sign of the numerator be changed. Hence we have

$$\frac{1}{a(a-b)(a-c)}=\frac{bc(b-c)}{abc(a-b)(a-c)(b-c)},$$

$$\frac{1}{b(b-a)(b-c)}=\frac{-1}{b(a-b)(b-c)}=\frac{-ac(a-c)}{abc(a-b)(a-c)(b-c)};$$

$$\frac{1}{c(c-a)(c-b)}=\frac{1}{c(a-c)(b-c)}=\frac{ab(a-b)}{abc(a-b)(a-c)(b-c)}.$$

Adding the numerators, their sum is the same as $(a-b)(a-c)(b-c)$; omitting this common factor, the result is $\frac{1}{abc}$.

The main point is to drill the pupil well in the law of the signs first; then the second important step is in the reduction and interpretation of fractional expressions. The laws of reduction are the same in principle as those in arithmetic; but the distinction between *number* and *quantity* ought to be preserved intact. Dissimilar algebraic expressions must frequently be reduced to equivalent, similar expressions, and in making reductions the learner should always keep in view the fact that it saves labor to change complex forms into simpler ones.

To fix principles in the mind, well-graded exercises must be chosen, and if the learner solves a problem in his own way and understands it thoroughly, and the teacher has a simpler solution, let him give it so that the pupil can see how it is solved. Let the teacher impress upon the pupil the similarity of the operations in arithmetic and algebra. Such teaching connects the two branches, and the one helps to explain the other.

Here, too, is a favorable opportunity for explaining the limitations of arithmetic.

In arithmetic, $\frac{0}{4}$ equals 0, $\frac{4}{0}$ equals 0, and $\frac{0}{0}$ equals 0. But in algebra the language is more general, and 0 is used in two senses: first, it may denote the absence of any value whatever, and it is then *absolute* zero; second, it is most frequently regarded as a very small quantity, and in this case it is said to be an *infinitesimal,* which signifies a quantity approaching zero, but zero is the limit which it never quite reaches. Since a quantity may be infinitely small, or infinitesimal, likewise a quantity may be infinitely great, and then it is called by writers *infinity.* Arithmetic, then, is limited by *zero* and *plus* infinity; while algebra deals

with negative and positive quantities running from *plus infinity* to *minus infinity.* An *infinitesimal* is represented by 0, and *infinity* by ∞.

Since a fraction may have any value whatever, if the denominator remains constant while the numerator continually decreases till it is less than any assignable value, or the numerator remains constant while the denominator decreases continually, we have the following equations, which must be translated into ordinary language:

$$\text{Let (1) } \frac{n}{d} = \text{any fraction;} \quad (2)\ \frac{0}{d} = 0; \quad (3)\ \frac{n}{0} = \infty.$$

Again, if either the numerator or the denominator remains constant while the other continually increases, or, if both decrease till each is less than any assignable value, there will result the following equations:

$$(4)\ \frac{n}{\infty} = 0; \qquad (6)\ \frac{0}{0} = ? \qquad (8)\ \infty - \infty = ?$$

$$(5)\ \frac{\infty}{d} = \infty; \qquad (7)\ \frac{\infty}{\infty} = ?$$

Under these forms, (5), (6), (7), and (8) require special attention. (5) shows that two infinities can be equal in one special case; otherwise unequal, and indeterminate; (6) is the symbol of indetermination; likewise (7) and (8) are indeterminate. The pupil very often rushes to a hasty conclusion, and decides that each infinity, or infinitesimal, is as large, or small, as any other like quantity.

There is another form that sometimes puzzles the learner, and owing to its importance in the higher departments of mathematics it is referred to here, and that is the *vanishing fraction.* To avoid ambiguity, the common factor must be suppressed in both numerator and denominator of a *vanishing fraction* before it is evaluated. In my opinion the teacher is not apt to err in giving too much attention

to fractions. The teaching here ought to be clear-cut and thorough.

Summarizing:

1. **Teach the definitions thoroughly.**

2. **Give special attention to the signs.**

3. All processes of **Reduction and Transformation must be thoroughly mastered.**

4. **Discussion of zero and infinity** as they occur in fractional forms.

Equations of the First Degree.

There are certain terms used in connection with the subject of Equations that the learner ought to know. A knowledge of them is indispensable to his successful progress in acquiring a clear conception of the subject. Mathematics has its language, which is precise and technical. Terms denote definite ideas, and their exact meaning must be learned as they occur to avoid confusion. To express this thought tersely,—*the technique of the science must be mastered.*

Points to be Emphasized.

1. **The exact definitions of the terms.**

2. **That the learner should get the exact meaning of a problem when he reads it, or when it is given to him for solution.**

3. **That the learner should think out the relation or relations existing between the known and unknown quantities, and express them in algebraic language. This is the thought process, and is called the statement of the problem.**

4. **The solution of the problem.**

The ability to express the conditions of a problem in equations depends upon the analytic power of the learner's mind. That frame of mind most conducive to a satisfactory investigation of mathematical relations is acquired and

stimulated chiefly through that mental characteristic—*abstraction.*

In thinking out the statement of a problem, the learner, from the very nature of the subject-matter, must concentrate his thoughts upon the conditions implied in the question, and exclude all extraneous matter.

Of course this act requires strong will power, and exercises of this character eminently qualify the learner for other operations of his reasoning faculties when applied to important affairs of life. Continuous thought is the difference between *a* man and *the* man.

After the statement of a problem is expressed, the learner must make up his mind how he ought to proceed to find the value, or values, of the unknown quantity. Sometimes it is much easier to state a problem than it is to solve it. A skillful algebraist foreseeing the difficulties likely to arise in resolving intricate expressions, will seek the simplest methods of expressing the conditions existing among the quantities, and the easiest way of finding their values. This suggestion resolves itself into two parts: 1. *Judgment in stating a problem.* 2. *Skill and tact in solving it.* Perhaps a generalized statement will include what is most important in this particular sphere of algebraic work,—**"sense enough to take advantage of short cuts."**

Dealing with Equations.

The first principle is that the equation, if not already, should be in its *simplest form;* that is, freed from fractions, terms transposed, and other necessary reductions performed. To effect what is here outlined, the principle of *Transposition* needs to be thoroughly taught. If the learner experiences any difficulty in understanding why a quantity changes its algebraic form when it is transposed from one member of the equation to the other, explanations by using figures instead of quantities generally remove all uncer-

tainty. Children can frequently think in numbers much better than in symbols of quantity. The individual idea precedes the general notion in intellectual unfoldment.

The steps in solving equations are in general the following:

1. To clear the equation of fractions, if necessary.

2. To transpose terms from one member of the equation to the other without destroying the value of the equation.

3. Collecting unknown quantities in the first member of the equation, and the known quantities in the second member.

4. Uniting similar terms.

5. Dividing through by the coefficient of the unknown quantity.

Equations of One Unknown Quantity.

Perhaps as much or more ingenuity is required to state *some* problems involving *one* unknown quantity than to express the relations existing among several unknown quantities. At first, the problem should be a special *one*, whether it is to be solved by the learner or by the teacher who illustrates for the benefit of the class. Many beginners can understand *special* or numerical problems, when the general ones confuse them. The mind appears impotent to make abrupt transitions, but gradual changes are effected easily. This indicates that the plan for the ordinary mind is, the *special problem* followed by the *generalized problem;* and probably this is the best method to pursue throughout the elementary course of study. The teacher in selecting problems for illustrating principles should choose simple ones. Principles are taught better and are fixed more firmly in the mind by simple illustrations than by complicated ones. A tangled skein is much harder to unwind than one which is free from kinks.

The solution should always be a model of condensed

neatness—clear enough to avoid all ambiguity, but sufficiently connected so as not to break the chain of reasoning at any point. A solution should be so plain that any one who understands mathematical language can read it off as easily as ordinary print.

Suppose we have the following to solve:

$$\begin{aligned}
&(1) \qquad 9x - 5x = 9 + 7;\\
&(2) \qquad 4x = 16;\\
&(3) \qquad x = 4.
\end{aligned}$$

The terms are simply collected, and (2) is divided by the coefficient of x.

In the problem $\left(\frac{a+b}{a-b}\right)x + x = \left(\frac{a+b}{a-b}\right) - 1$, we have to transform, reduce, and divide through by the coefficient of x.

Insist upon the pupil's marking each step in the solution.

The teacher must introduce one difficulty only at a time. A child learns to walk by taking one step after another,—not two or three steps at once. So in teaching, one thing well taught, and then onward to another, keeping the dependence and connection in solid ranks. Straggling teaching is one of the banes of school-work.

There is a tendency among pupils studying Algebra to pass away from the domain of Arithmetic, and to separate the two subjects so widely that no close connection can be preserved. Such a view, in my opinion, weakens both algebraic and arithmetical skill. As a remedy, the teacher should insist upon the learner's solving certain special problems arithmetically as well as algebraically. Algebraic problems solved by Arithmetic enable the learner to compare the two methods, and to decide upon the merits of both. The ability to do a thing in several different ways is a strong presumption in favor of "clear-cut knowledge."

Two or More Unknown Quantities.

As long as an equation contains two unknown quantities and there is no way of getting rid of either of these quantities except by imposing an arbitrary value on one of them, the equation is *indeterminate.* There must be as many independent equations as there are unknown quantities. Combining two independent equations that are satisfied by the same values of the unknown quantities in such a manner as to obtain a single equation having but one unknown quantity, constitutes the process called *Elimination.* Notwithstanding how many simultaneous equations there may be in a problem, the process of reduction, by whatever method, is to find one equation containing one unknown quantity, and from which the value of this quantity can be expressed in known terms. Two equations must first be transformed into *one;* three into two, and the two into one; four into three, three into two, and two into one, and so on.

Elimination.

Elimination by Addition or Subtraction is the first method acquired by the learner. After it is thoroughly understood, he is ready to begin Elimination by Comparison. A good plan for him is to solve two equations by Addition or Subtraction, and then to solve the same by Comparison. Let him work both ways till he can solve equations by either method with equal facility.

Next, he is to learn how to eliminate by Substitution. Knowing these three methods equally well, he is prepared to use whichever appears most convenient. Sometimes one method in a given case is far preferable to either of the others. To impress this truth, let the learner solve a problem by all three methods and then compare processes.

The Form of Solutions.

Mathematical studies afford excellent opportunities for teaching exactness in written composition. All work, whether on paper, slate, or blackboard, should be concise yet elegant in style, capital letters properly placed, and punctuation points correctly used. The work in all cases ought to be ready to set in type without any corrections whatever. Slovenly work denotes slipshod habits of body and mind. First-class work is done by first-class workers. To know what good work is, compare the methods of solving the same or similar problems in our best text-books.

While the form of the solution is much, yet it is not the chief object. It signifies that the operator knows how to express himself in an artistic manner, and in language that is easily read and interpreted. To secure the best possible results, the learner or class must be led gradually upward and onward, mastering each new difficulty as it occurs. The teaching must be so done that the members of the class never lose confidence, each in his own ability. Piling on too much and that which is too difficult, discourages persons of average zeal; whereas if the work is properly parcelled out, the working spirit is kept intact.

Other Methods of Elimination.

There are two other methods of Elimination which may be conveniently employed by those well skilled in Algebra. The first of these methods is that of *Undetermined Multipliers*. It consists in the introduction of a factor to which a definite value is assigned before the solution is completed. Many algebraists claim superior advantages for this method of Elimination in solving literal equations. As this method

is not usually included in elementary works, two problems will be solved to illustrate it.

$$
\begin{array}{lll}
\text{Given} & (1) & x + y = 15, \\
 & (2) & x - y = 7, \quad \text{to find } x \text{ and } y. \\
(3) = m(1), & & mx + my = m15. \\
(4) = (3) - (1), & & x(m-1) + y(m+1) = m15 - 7.
\end{array}
$$

Put $m - 1 = 0$, $\therefore$ $m = 1$.

Substitute the value of m in (4), and it becomes

$$2y = 8, \quad \text{and} \quad y = 4; \quad \text{and} \quad x = 11.$$

Instead of taking $m - 1 = 0$, we can use $m + 1 = 0$, and then find x, and afterwards y.

$$
\begin{array}{lll}
\text{Given} & (1) & ax + by = c, \\
 & (2) & a'x + b'y = c', \quad \text{to find } x \text{ and } y. \\
(3) = m(1), & & max + mby = mc. \\
(4) = (3) - (1), & & (ma - a')x + (mb - b')y = mc - c'.
\end{array}
$$

Put $ma - a' = 0$, $\therefore$ $m = \dfrac{a'}{a}$; consequently (4) becomes

$$(5) \qquad \left(\frac{a'b}{a} - b'\right)y = \frac{a'c}{a} - c';$$

$$(6) = (5), \qquad y = \frac{a'c - ac'}{a'b - ab'} = \frac{ac' - a'c}{ab' - a'b}.$$

Whence

$$x = \frac{bc' - b'c}{a'b - ab'}.$$

Have the learner point out the differences and the similarities in the numerators of x and y. What letters are permuted?

Should we desire to extend this method of solution to

three or more unknown quantities, it leads to some beautiful results.

Let it be required to solve the following equations:

$$(1)\qquad A'x + B'y + C'z = D',$$
$$(2)\qquad A^2x + B^2y + C^2z = D^2,$$
$$(3)\qquad A^3x + B^3y + C^3z = D^3.$$

$$(4) = m(1),\quad mA'x + mB'y + mC'z = mD';$$
$$(5) = n(2),\quad nA^2x + nB^2y + nC^2z = nD^2;$$
$$(6) = (4) + (5),$$
$$(mA' + nA^2)x + (mB' + nB^2)y + (mC' + nC^2)z = mD' + nD^2 - D^3;$$
$$(7) = (6) - (3),$$
$$(mA' + nA^2 - A^3)x + (mB' + nB^2 - B^3)y + (mC' + nC^2 - C^3)z = mD' + nD^2 - D^3.$$

To cause y and z to vanish from (7),

$$mB' + nB^2 - B^3 = 0, \quad \text{and} \quad mC' + nC^2 - C^3 = 0,$$

whence

$$x = \frac{mD' + nD^2 - D^3}{mA' + nA^2 - A^3}. \qquad (8)$$

The values of m and n must be found from

$$mB' + nB^2 - B^3 = 0, \quad \text{and} \quad mC' + nC^2 - C^3 = 0,$$

whence

$$m = \frac{B^3C^2 - C^3B^2}{B'C - C'B^2},$$

and

$$n = \frac{B'C^3 - B^3C'}{B'C^2 - B^2C'}.$$

Substituting the values of m and n in (8), we have

$$(9) = (8), x = \frac{D'B^2C^3 + D^2B^3C' + D^3B'C^2 - D'B^3C^2 - D^2B'C^3 - D^3B^2C'}{A'B^2C^3 + A^2B^3C' + A^3B'C^2 - A'B^3C^2 - A^2B'C^3 - A^3B^2C'}.$$

Proceeding in a similar manner, the values of y and z are found to be

$$(10) \qquad y=\frac{A'D^2C^3+A^2D^3C'+A^3D'C^2-A'D^3C^2-A^2D'C^3-A^3D^2C'}{A'B^2C^3+A^2B^3C'+A^3B'C^2-A'B^3C^2-A^2B'C^3-A^3B^2C'},$$

and

$$(11) \qquad z=\frac{A'B^2D^3+A^2B^3D'+A^3B'D^2-A'B^3D^2-A^2B'D^3-A^3B^2D'}{A'B^2C^3+A^2B^3C'+A^3B'C^2-A'B^3C^2-A^2B'C^3-A^3B^2C'}.$$

Examining these three results, the denominator is the same in each,—composed of three positive products, each product being composed of three factors; and also three negative products of three factors each. The letters composing these factors are alphabetically arranged, and in the first product the letters have exponents corresponding to their order in the alphabet; thus $A'B^2C^3$ have 1, 2, 3, respectively. If we add one to each of these exponents, and write 1 wherever the sum of $3+1$ equals 4, we shall have the exponents of the second product of the denominator as follows: 2, 3, 1; and for the third product, 3, 1, 2.

For the negative products, after the first is obtained, whose exponents are 1, 3, 2, the others are derived by adding 1 as in the case of the positive factors. Thus, 2, 1, 3, and 3, 2, 1. Again, the numerator of (9) is the same as the denominator, if D be written for A. Hence a simple permutation gives the numerator of x.

The numerator of (10) is found by writing D for B in the common denominator, retaining the exponents; and the numerator of (11) found by writing D for C, observing the same restrictions.

The next step is to show how the common denominator can be obtained in an easy manner by dealing exclusively with the coefficients of x, y, and z. Let us write the coefficients and the absolute terms in the order they occur in equations (1), (2), (3); we have—

$$\begin{array}{llll}
(1) & A' & B' & C' = D'\,; \\
(2) & A^2 & B^3 & C^2 = D^2\,; \\
(3) & A^3 & B^3 & C^3 = D^3\,; \\
(1) & A' & B' & C' = D'\,; \\
(2) & A^2 & B^2 & C^2 = D^2\,; \\
(3) & A^3 & B^3 & C^3 = D^3.
\end{array}$$

I duplicate the equations in order to make the explanation perfectly clear. In the first set of equations, if we begin at A' and pass to B^2 and thence to C^3 obliquely downward, we have $A'B^2C^3$, the first product of the common denominator. For the second, begin with A^2, thence to B^3, thence to C', we have the second product, A^2B^3C'; and beginning with A^3, thence to B' in the second set, and thence to C^2 in the second set, we have $A^3B'C^2$, the third product.

To find the negative products, we count obliquely upward, beginning with A' in the second set of equations; thus, $A'B^3C^2$. Next, with A^2, and we have $A^2B'C^3$; and for the third product, A^3B^2C'. Permuting as previously explained, we readily get the numerators from the common denominator. Or we may proceed thus:

Write D instead of A in the original equations, and we have these expressions:

$$\begin{array}{llll}
(1) & D' & B' & C'\,; \\
(2) & D^2 & B^2 & C^2\,; \\
(3) & D^3 & B^3 & C^3\,; \\
(1) & D' & B' & C'\,; \\
(2) & D^2 & B^2 & C^2\,; \\
(3) & D^3 & B^3 & C^3.
\end{array}$$

Beginning with D and moving obliquely downward, we have

$$D'B^2C^3 + D^2B^3C' + D^3B'C^2 - D'B^3C^2 - D^2B'C^3 - D^3B^2C',$$

which is the numerator of x.

To find y, write D instead of B, thus:

$$\begin{array}{llll}
(1) & A' & D' & C'; \\
(2) & A^2 & D^2 & C^2; \\
(3) & A^3 & D^3 & C^3; \\
(1) & A' & D' & C'; \\
(2) & A^2 & D^2 & C^2; \\
(3) & A^3 & D^3 & C^3.
\end{array}$$

Proceeding as before, we have

$$A'D^2C^3 + A^2D^3C' + A^3D'C^2 - A'D^3C^2 - A^2D'C^3 - A^3D^2C'$$

for the numerator of y.

To find the numerator of z, write D for C, and proceed as above.

In solving numerical equations particular attention must be given to the algebraic signs of the coefficients.

This method of elimination is the beginning of what is called "Determinants" in the Higher Analysis, and its applications are so far-reaching that additional exercises are given in order to explain its uses.

Given $\quad Ax + By = C,$
$\quad Dx + Ey = G,$ to find x and y.

By any of the usual methods of elimination,

$$x = \frac{BG - CE}{AE - BD}, \quad \text{and} \quad y = \frac{AG - DC}{AE - BD}.$$

To solve by the foregoing method, we have

$$\begin{array}{ll}
A & B = C, \\
D & E = G.
\end{array}$$

Beginning with A, we get AE for the positive product of the common denominator; $-DB$ for the negative product. Also for the numerator of x we have BG positive and

EC negative. For the numerator of y, AG positive, DC negative.

Given (1) $6x + 5y = 61$,
(2) $3x + 4y = 38$, to find x and y.

Therefore

$$x = \frac{61 \times 4 - 38 \times 5}{6 \times 4 - 3 \times 5} = 6, \text{ and } y = \frac{6 \times 38 - 3 \times 61}{6 \times 4 - 3 \times 5} = 5.$$

Again, $\left\{\begin{array}{l} 2x + 4y = 12 \\ 3x - 2y = 10 \end{array}\right\}$, to find x and y.

Here

$$x = \frac{12 \times -2 - 10 \times 4}{2 \times -2 - 3 \times 4} = 4, \text{ and } y = \frac{2 \times 10 - 3 \times 12}{2 \times -2 - 3 \times 4} = 1.$$

Let it be required to find the values of x, y, and z in the following:

$$\begin{array}{ll} (1) & 3x + 4y - 2z = 10; \\ (2) & 5x - 2y + 3z = 16; \\ (3) & 4x + 2y + 2z = 22. \end{array}$$

Solution.

$$\begin{array}{l} 3 + 4 - 2 = 10; \\ 5 - 2 + 3 = 16; \\ 4 + 2 + 2 = 22; \end{array}$$

whence

$$x = \frac{10 \times -2 \times 2 + 16 \times 2 \times -2 + 22 \times 4 \times 3 - 10 \times 2 \times 3 - 16 \times 4 \times 2 - 22 \times -2 \times -2}{3 \times -2 \times 2 + 5 \times 2 \times -2 + 4 \times 4 \times 3 - 3 \times 2 \times 3 - 5 \times 4 \times 2 - 4 \times -2 \times -2} = 2.$$

Since the denominator, -58, is common, to find y we proceed as follows:

$$\begin{array}{l} 3 + 10 - 2; \\ 5 + 16 + 3; \\ 4 + 22 + 2; \end{array}$$

whence

$$y = \frac{96 - 220 + 120 - 198 - 100 + 128}{-58} = 3,$$

and

$$z = \frac{-132 + 100 + 256 - 96 + 80 - 440}{-58} = 4.$$

When the equations are symmetrical, the simplest solution is to find the value of one unknown quantity and then permute for the values of the others. Much time and labor can be saved in the solutions of equations if the instructor will drill his classes thoroughly in handling symmetrical expressions.

Reference has already been made to that branch of mathematics called "Determinants." As one of the most valuable lessons that a teacher ever imparts is that of stimulating his pupils to higher endeavors, so a glimpse of what lies beyond frequently opens out a new prospect to a pupil and gives him a clearer perspective of what is vague and indistinct in his mind. The power transmitted, and a burning desire to learn more, are the fruits of good teaching; and he who does not leave his pupils with these traits of character deeply impressed, is not a success in the educational work. Consequently, with what has just been given, the ambitious teacher can instruct his pupils in Elimination as taught in Elementary Treatises on *Determinants.*

Thus, $\left\{ \begin{array}{r} x + 3y = 10 \\ 3x + 2y = 9 \end{array} \right\}$, to find x and y.

$$\therefore\ x = \frac{\begin{vmatrix} 10 + 3 \\ 9 + 2 \end{vmatrix}}{\begin{vmatrix} 1 + 3 \\ 3 + 2 \end{vmatrix}} = \frac{10 \times 2 - 9 \times 3}{1 \times 2 - 3 \times 3} = \frac{20 - 27}{2 - 9} = 1,$$

and

$$y = \frac{\begin{vmatrix} 1 + 10 \\ 3 + 9 \end{vmatrix}}{\begin{vmatrix} 1 + 3 \\ 3 + 2 \end{vmatrix}} = \frac{1 \times 9 - 3 \times 10}{1 \times 2 - 9 \times 1} = \frac{9 - 30}{2 - 9} = 3.$$

This solution is performed in precisely the same manner as the preceding under this heading, but is more condensed.

In Determinants it would be solved thus:

$$x = \frac{\begin{vmatrix} 10 & 3 \\ 9 & 2 \end{vmatrix}}{\begin{vmatrix} 1 + & 3 \\ 3 & 2 \end{vmatrix}} = 1.$$

The cross multiplication being performed mentally, the operation is greatly abridged.

As a second exercise the following is selected:

$$\left\{\begin{array}{l} 3x + 4y = 18 \\ 2x - y = 1 \end{array}\right\}, \text{ to find } x \text{ and } y.$$

$$x = \frac{\begin{vmatrix} 18 & 4 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}} = 2; \qquad y = \frac{\begin{vmatrix} 3 & 18 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}} = 3.$$

In my own experience I found that those studying any branch of mathematics were always stimulated to greater exertion if glimpses were given them of what lay beyond their immediate knowledge, especially if the newer method contracted the labor of that the student already knew. True teaching is not measured so much by the amount of information or number of facts communicated as by the constant impetus given to the mind, by which it is carried forward ever after in pursuit of knowledge. Of late our Modern Algebra has been so extended, and in so many respects simplified, that newer methods of handling equations should occupy a portion of the time usually devoted to this subject in our common and high schools. In other words, the teacher should be two or three sizes larger than the subject he teaches.

Evolution.

In Arithmetic the pupil learned how to extract the square root and the cube root of numbers, and perhaps his teacher gave him an insight into the working of Horner's Method in extracting higher numerical roots. The pupil saw that in Involution he was required to multiply quantities continually. Involution, then, is an act of involving a quantity, or shrouding it in obscurity. Evolution is just the reverse. It is to unroll or to unfold, and to make clear. Involution multiplies and obscures: evolution *un*-multiplies. Should I say to a friend, "I have squared or cubed any quantity, and this is its cube; tell me the quantity," he takes the complex product, and from it finds the quantity required: this is evolution.

The arithmetical treatment of this subject is quite a help to the learner. It is a forerunner, and clears the way. In fact, no great progress can be made in Algebra without a thorough knowledge of Involution and Evolution as treated of in the text-books on Arithmetic.

Knowledge is so related that the skillful teacher will always use the simpler kinds to unravel the more complex.

The essential points to be emphasized are:

1. The learner should examine the quantity carefully for the purpose of discovering whether it is a perfect power.

2. It may be a perfect power of one degree but not of another degree.

3. The algebraic sign of the entire expression must be kept constantly in mind. Look, think, and remember.

4. The sign of each term.

5. The sign of the odd roots of any quantity;

as, $\sqrt[3]{a} = a$, and $\sqrt[3]{-a} = -a$.

6. The sign of the even roots of any quantity;

as, $\sqrt{a^2} = a$ or $-a$; that is, $\pm a$.

Again, $\sqrt[6]{a^6} = \pm a$.

7. The even root of a negative quantity is impossible. Why?

8. The *mnth* root of a quantity is equal to the *mth* root of the *nth* root of that quantity.

Thus, $\sqrt[4]{a} = \sqrt{\sqrt{a}}$; $\sqrt[6]{a} = \sqrt[3]{\sqrt{a}}$.

Evolution of Polynomials.

The works on Algebra give methods of extracting roots, and nearly all of them are closely connected with the ordinary methods taught in Arithmetic. After drilling a class thoroughly in the ordinary method of performing this work, either in Arithmetic or Algebra, I am fully satisfied from my own experience that the pupil or class should be thoroughly drilled in "Horner's Method." In every solution of numerical equations above the second degree, unless the root can be discovered by inspection, it is the shortest and easiest way to do the work; and in the solution of numerical algebraic equations it is the only method that is used, unless an equation is resolved by some antiquated process as a mere matter of curiosity. The excellence of Horner's Method consists in this, that it can be employed as easily in one degree as in another, and the work preceding in a solution helps what is to follow. The odd root can be as easily extracted as the even root.

For instance, let it be required to extract the fifth root of

$$32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1.$$

Col. 1.	Col. 2.	Col. 3.	Col. 4.
$2x$	$4x^2$	$8x^3$	$16x^4$
$4x$	$12x^2$	$32x^3$	$\underline{80x^4}$
$6x$	$24x^2$	$\underline{80x^3}$	$80x^4 - 80x^3 + 40x^2 - 10x + 1.$
$8x$	$\underline{40x^2}$	$80x^3 - 40x^2 + 10x - 1.$	
$\underline{2x}$	$40x^2 - 10x + 1.$		
$10x - 1.$			

$$
\begin{array}{l|l}
32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1 & \underline{2x - 1.} \\
\underline{32x^5} & \\
\qquad - 80x^4 + 80x^3 - 40x^2 + 10x - 1 & \\
\qquad \underline{- 80x^2 + 80x^3 - 40x^2 + 10x - 1} & \\
\qquad\qquad\qquad 0 &
\end{array}
$$

Remark. If the pupil does not know how to extract roots by Horner's Method, now is an excellent opportunity to teach him.

While this is a special case, yet it will serve as a guide for any numerical equation, whether arithmetical or algebraic.

Involution and Evolution.

There are certain notions in regard to *Involution* and *Evolution* that must be thoroughly fixed in the learner's mind at this time. He has learned thus far that quantities can be added, subtracted, multiplied, and divided, and now he is ready to take two additional steps, namely, that quantities can be raised to any power, or have their roots extracted.

The things to be critically distinguished are:

1. The Base or Root.
2. The Exponent of the Power.
3. A Perfect Power.
4. An Imperfect Power.
5. The Sign of the Power.
6. The Sign of the Root.
7. The nth Power of a Product.
8. The nth Root of a Power.
9. The Coefficient of a Power.
10. All Even Powers of a Quantity.
11. All Odd Powers of a Quantity.

The next subject requiring attention is the treatment of monomials. Since a monomial may have any number or

known quantity for a coefficient, the learner must observe particularly the sign of the monomial, the sign of the coefficient of the power, and the sign of the exponent.

Suppose the quantity to be $2a$ which is to be raised to the nth power.

Then $$(2a)^n = 2^n a^n.$$

If $n = 2$, we have
$$(2^n a^n) = 2^2 a^2 = 4a^2.$$

If $n = 3$, it becomes
$$(2^n a^2) = 2^3 a^3 = 8a^3 = 2a \times 2a \times 2a.$$

If $2a$ be negative by actual multiplication, then

$$-2a \times -2a = 4a^2,$$

but $-2a \times -2a \times -2a = -8a^3$,

and $-2a \times -2a \times -2a \times -2a = 16a^4$, and so on.

That is, inductively, **All powers of a positive monomial are positive**; and, **All even powers of a negative monomial are positive, and all odd powers are negative.**

Questions to be Answered or Found Out by the Learner.

$(a^2)^3 =$ what? $(a^3)^2 =$ what? $(a^m)^n =$ what?

If $m = 4$, $n = 3$, $a = 2$, find the value of $(a^m)^n$.

Is $(a^m)^n = a^{m+n}$? Why? If not, what should it be?

What is the difference between $3a^{m\int n}$ to the mth power, and $3a^{m\int n}$ to the nth power?

Polynomials.

It has previously been shown how *to square* $(a + b)$ and $(a - b)$. This process may be extended so as to include any number of terms without performing the actual work.

Thus,
$$(a + b)^2 = a^2 + 2ab + b^2,$$
and
$$(a - b)^2 = a^2 - 2ab + b^2.$$

Again,

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc,$$

and

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad \\ +2bc+2bd+2cd.$$

An examination of the results will indicate the law observed in writing the complete product; hence to square a polynomial,—*Add to the square of each term twice the product of that term and every term that follows it.*

To find $(a+b-c)^2$, we simply change $a^2+b^2+c^2+2ab+2ac+2bc$ to $a^2+b^2+c^2+2ab-2ac-2bc$.

By changing the sign of any letter a new formula is obtained; but it must be remembered that the even power of a negative quantity is positive. Again, in these expansions the teacher should not fail to direct the learner's attention to the ease with which polynomials can be raised to higher powers without recourse to actual multiplication. Laws stand far above all empirical processes.

$$(a+b)' = a+b;$$
$$(a+b)^2 = a^2+2ab+b^2;$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3;$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4;$$
$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^2+5ab^4+b^5.$$

The learner will observe the following particulars:

1. The leading letter decreases its power regularly from the left toward the right, while the other increases regularly.

2. The sum of the exponents in each term is constant.

3. The coefficient of the first term is one, and the coefficient of the second term is the exponent of the binomial.

4. The coefficient of any term is found by multiplying the coefficient of the preceding term by the exponent of the

leading letter of that term, and then dividing the product by the number of the term counting from the left.

5. If the binomial be of the form $(a - b)$, then in the expansion the odd terms will be positive, and the even terms negative, counting from the left.

Again, if we observe the coefficients in the expansions, we have the following:

(1) $a + b = 1 + 1 = 2;$
(2) $(a + b)^2 = 1 + 2 + 1 = 4;$
(3) $(a + b)^3 = 1 + 3 + 3 + 1 = 8;$
(4) $(a + b)^4 = 1 + 4 + 6 + 4 + 1 = 16;$
(5) $(a + b)^5 = 1 + 5 + 10 + 10 + 5 + 1 = 32;$
(6) $(a + b)^6 = 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64;$
(7) $(a + b)^7 = 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128.$

Hence, the sum of the coefficients forms a geometrical progression whose constant ratio is 2; but if either a or b be negative, then the sum of the coefficients in each expansion is *zero*.

Another property of the coefficients is still more remarkable than any yet mentioned. For instance, if we regard the first and last coefficients in each expansion to be *one*, then we can derive the other coefficients in the following simple manner, thus:

For $(1) = 1 + 1;$
and
$$(2) = 1 + 2 + 1;$$
that is, we add the coefficients $(1 + 1)$ of (1), which sum is the middle coefficient of (2).

Again, adding $(1 + 2)$ and $(2 + 1)$, we have
$$(3) = 1 + 3 + 3 + 1.$$

Now, $(1 + 3)$, $(3 + 3)$, $(3 + 1)$, and we have
$$(4) = 1 + 4 + 6 + 4 + 1;$$
for
$$(5) = 1 + 5 + 10 + 10 + 5 + 1,$$
and so on.

From this simple law the learner can write the expansion without any difficulty.

Or it may be obtained thus:

$$(1) = 1 + 1;$$

$$(2) = \begin{array}{l} 1 + 1 \\ \quad\quad 1 + 1 = 1 + 2 + 1; \end{array}$$

$$(3) = \begin{array}{l} 1 + 2 + 1 \\ \quad\quad 1 + 2 + 1 = 1 + 3 + 3 + 1; \end{array}$$

$$(4) = \begin{array}{l} 1 + 3 + 3 + 1 \\ \quad\quad 1 + 3 + 3 + 1 = 1 + 4 + 6 + 4 + 1; \end{array}$$

$$(5) = \begin{array}{l} 1 + 4 + 6 + 4 + 1 \\ \quad\quad 1 + 4 + 6 + 4 + 1 = 1 + 5 + 10 + 10 + 5 + 1; \end{array}$$

$$(6) = \begin{array}{l} 1 + 5 + 10 + 10 + 5 + 1 \\ \quad\quad 1+5+10+10+5+1=1+6+15+20+15+6+1. \end{array}$$

There is another modification of expansion when the polynomial consists of more than two terms which requires something more than a passing remark. The case is this: Suppose it be required to expand $(a + b + c)^3$. By actual multiplication it can be shown that

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2a + 3b^2c + 3c^2a + 3c^2b + 6abc.$$

Also,

$$(a + b + c + d)^3 = a^3 + b^3 + c^3 + d^3 + 3a^2b + 3a^2c + 3a^2d + 3b^2a + 3b^2c + 3b^2d + 3c^2a + 3c^2b + 3c^2d + 3d^2a + 3d^2b + 3d^2c + 6abc + 6abd + 6acd + 6bcd.$$

The law is: **Write the cube of each term, plus three times the square of each term into each of the other terms, and plus six times the product of all the terms, taken three at a time.**

A little practice and attention will enable any one to write without hesitancy the cube of any polynomial. The advantage of understanding how to employ "short methods

and to permute quantities in symmetrical expressions" will save both time and labor. Practice and theory are thus combined, and the theory is turned to practical account.

With many the mental effort is not too great in expanding directly such expressions as

$$(2a + 3b)^3, \quad (3a^2 - 4b^2)^3, \quad (6a^2x + 7b^3y)^3,$$

and so on.

With advanced classes, or those of mature judgment, the teacher may very profitably call attention to the expansion of $(a \pm b)^n$, and show how to apply it when $n = 2$, $n = 3$, $n = 4$, $n = 5$, and so on; or even when n is fractional.

Such glimpses open the way to a better understanding of Newton's Formula than any mere abstract statement could possibly do. The very idea that the learner is going over the same line that Sir Isaac Newton once pursued will have an inspiriting effect.

Radical Quantities.

Exponents.

The Theory of Exponents deserves more than a passing notice. Thus far the pupil has used integral positive exponents. Since any quantity may be regarded as positive or negative, integral or fractional, so an exponent may be treated under any two of these four conditions.

Things to Teach.

1. Teach clearly that the numerator of the exponent indicates the power to which the quantity is to be raised, and the denominator denotes the root to be extracted. Thus, $a^{\frac{2}{3}} = \sqrt[3]{a^2}$ denotes that a is to be "squared," and then the "cube root" extracted.

Also, that

$$a^{\frac{1}{2}} = \sqrt{a}, \qquad a^{\frac{1}{3}} = \sqrt[3]{a}, \qquad a^{\frac{3}{4}} = \sqrt[4]{a^3}, \qquad a^{\frac{n}{m}} = \sqrt[m]{a^n},$$

are applications of this principle.

2. The meaning of the following and similar exercises:

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a;$$

$$a^{\frac{1}{2}} \times a^{\frac{3}{2}} = a^{\frac{1}{2}+\frac{3}{2}} = a^2;$$

$$a^{\frac{1}{2}} \times a^{\frac{2}{3}} = a^{\frac{3}{6}+\frac{4}{6}} = a^{\frac{7}{6}} = \sqrt[6]{a^7};$$

$$a^{\frac{1}{2}} \times a^{\frac{1}{3}} \times a^{\frac{1}{4}} = a^{\frac{6}{12}+\frac{4}{12}+\frac{3}{12}} = a^{\frac{13}{12}} = \sqrt[12]{a^{13}}.$$

3. $$a^{\frac{n}{d}} = \sqrt[d]{a^n}; \quad a^{\frac{n}{d}} \times a^{\frac{n}{2d}} = a^{\frac{2n+n}{2d}};$$

$$a^{\frac{n}{d}} \times a^{\frac{m}{p}} \times a^{\frac{r}{\,}} = a^{\frac{nps+mds+rdp}{dps}}.$$

That is, the exponents are reduced to a common denominator, and added.

4. $$a^{-2} = \frac{1}{a^2}; \quad a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}}; \quad a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}.$$

While many expressions having negative exponents frequently occur, yet the reciprocals of these expressions can be taken, and then the negative exponents become positive, and are so treated. However, such quantities can be handled quite as easily as those having positive exponents.

Quantities having negative exponents may be regarded as having their origin in a fractional expression; thus, $a^{-\frac{1}{2}}$ may be conceived as arising from

$$\frac{a}{a^{\frac{3}{2}}} = a^{1-\frac{3}{2}} = a^{\frac{1}{2}} = \frac{1}{a^{\frac{3}{2}}}; \quad \text{or} \quad \frac{a^2}{a^{\frac{5}{2}}} = \frac{1}{a^{\frac{1}{2}}} = a^{-\frac{1}{2}};$$

or from any other form in which the denominator is $a^{\frac{1}{2}}$ greater than the numerator.

Exponents convey a language of deep significance in mathematics, and are so important that one mistake in them will vitiate a whole discussion or demonstration. Mathematics prides itself on the brevity of its language and on the universality of its symbols. Yet every symbol must be

interpreted exactly. In one sense, exponents are instruments employed by algebraists to reduce heterogeneous, or refractory, quantities to homogeneous ones whenever it is possible. However, no new principle is employed in treating exponents that has not been already used. Exponents are the "handspikes of Algebra," and by the aid of which dissimilar quantities may be grouped, separated, multiplied, or divided.

Here, again, the learner's algebraic vocabulary must be enlarged by the introduction of a few new terms. Therefore it is necessary that he should learn the following for all time, namely:

1. A Simple Radical Quantity: as, $3\sqrt{8}$.
2. The Radical Factor: $\sqrt{8}$.
3. The Radical Coefficient: 3.
4. The Degree of the Radical.
5. When a Radical is in its Simplest Form.
6. A Radical Quantity.
7. An Irrational Quantity.
8. Similar Radicals.
9. Dissimilar Radicals.

Reduction of Radicals.

The Reduction of Radicals consists in changing their forms without altering values. For instance, it may be more convenient to change a complicated expression into a simpler one, or to render dissimilar expressions similar, or to change irrational quantities into rational ones.

After knowing how to read Radicals, the simplest process is to put a quantity under the radical sign, or to take a quantity from under the radical sign. Simple exercises for purposes of illustration are always preferable, because less confusing to the pupil.

Before proceeding further with this subject, the teacher should call attention to the *method of reading* exponents. Thus, "a^d" is read, the *d*th *power* of *a*, or *a* to the *d*th *power*, if *d* is a positive integer; but if *d* be integral and negative, then it should be read, "reciprocal of the *d*th *power* of *a*," or the reciprocal of "*a*" to the *d*th *power*. If *d* be fractional, say, as "$\frac{2}{3}$," it should be read "*a exponent d*," "or *a exponent* $\frac{2}{3}$;" but never "*two-thirds power*."

1. Familiarize the pupil with the process of putting coefficients under the radical sign. A few exercises will be sufficient for this purpose. Then, let him take the same quantity out from under the sign. Select problems from the text. Practice makes perfect. Teach each step before advancing to the next.

2. When the quantity under the radical sign is a common fraction, change it to an equivalent expression in which the quantity under the radical sign shall be entire. This is an invaluable transformation.

3. Change radical quantities to their simplest forms, or reduce a radical quantity to a higher or lower degree. These reductions correspond to certain lower arithmetical operations which the pupil will readily perceive. Lastly, reduce radicals having unequal indices to equivalent quantities having the same indices.

These three steps have for their object the preparation of radical quantities for addition, subtraction, multiplication, and division.

Require the learner to show upon what principle, definition, or axiom each step depends in the solution of a problem. This is one of the very best mental disciplines. To follow out consecutively every principle, and trace it backward to its origin or dependence, cultivates a close habit of thought that will be invaluable in after life. *It is to unroll the scroll.*

As an illustration of this suggestion let it be required—
To find the sum of

$$a\left(1+\frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right)^{\frac{1}{2}} \quad \text{and} \quad b\left(1+\frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}\right)^{\frac{1}{2}}.$$

Solution.

$$a\left(1+\frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right)^{\frac{1}{2}} = a\left(\frac{a^{\frac{2}{3}}+b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right)^{\frac{1}{2}} = \left(\frac{a^{\frac{8}{3}}+a^2b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right)^{\frac{1}{2}} =$$

$$\left(a^2+a^{\frac{4}{3}}b^{\frac{2}{3}}\right)^{\frac{1}{2}} = a^{\frac{2}{3}}\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}.$$

Similarly,

$$b\left(1+\frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}\right)^{\frac{1}{2}} = b^{\frac{2}{3}}\left(b^{\frac{2}{3}}+a^{\frac{2}{3}}\right)^{\frac{1}{2}}.$$

Whence,

$$(a^{\frac{2}{3}}+b^{\frac{2}{3}})(b^{\frac{2}{3}}+a^{\frac{2}{3}})^{\frac{1}{2}} = \{(a^{\frac{2}{3}}+b^{\frac{2}{3}})^3\}^{\frac{1}{2}}.$$

The first step is to reduce the first quantity to an improper fraction. This depends upon what principle? The second, to introduce a coefficient under the radical sign. Upon what does it depend? What operation is involved? How is it done? Third, to suppress a common factor. Why? What principle is involved?

Fourth, to take a factor from under the radical sign. How is it done? Why is the value of the expression not changed?

Fifth, to multiply two factors dissimilarly involved and to indicate their product.

What axioms are involved in all these operations? Name them. How many of the original mathematical operations are involved in this solution?

In the solution of all Radical Quantities let these truths be deeply impressed on the mind of the learner,—*that dissimilar quantities must, if possible, be made similar; that*

complicated expressions are to be put into simpler or simplest forms; that quantities, so far as possible, must be freed from indicated roots; and that irrational quantities are to be rationalized.

To enforce these truths when a problem is solved, the teacher will first call attention to each step as it is taken, and afterwards have the pupil or class explain. Skill in Algebra is secured by intelligent practice. If a boy is obliged to roll a heavy log which he cannot lift, he uses his inventive faculties to assist him. Handspikes and props are found, and he makes a practical use of the lever. So it should be in attacking a mathematical problem. The knowledge the learner already has of definitions, axioms, principles, and of operations previously employed are now laid under tribute to help him solve the questions proposed. In the case of the log, the boy knew exactly what he had to do; but what is intended in the enunciation of a problem may not be quite so evident to the reader, hence his first mental effort is to ascertain what is required; or to bring the proposed question under some form with which he is already familiar.

There are two more points under Radicals that require attention. I refer to Quadratic Surds and Imaginary Quantities.

Under the first, aside from methods of solution, the following principles lie at the foundation of the subject, namely:

1. *That no irrational quantity may be expressed by a fraction.* Why?

2. A quadratic surd cannot be equal to the sum of a rational quantity and a quadratic surd. Why?

3. If two quadratic surds cannot be made similar, their product is irrational. Why?

4. A quadratic surd cannot be equal to the sum of two dissimilar quadratic surds. Why?

5. In an equation of the form $x \pm \sqrt{y} = a \pm \sqrt{b}$, $x = a$, and $\sqrt{y} = \sqrt{b}$. That is, the rational parts are equal, and also the irrational parts are equal. Required, the proof.

In the treatment of "Imaginary Quantities," the apparent deviation from ordinary rules of procedure appears in the case of Multiplication. Algebraists have resorted to various expedients in order to explain the apparent ambiguity with more or less success. The questions that shake the learner's faith are these: If a quantity is imaginary, how can it become a real quantity? Is there any way to tell how, for instance, the pupil shall know $\sqrt{a^2} = -a$, unless he knew beforehand that *a is minus* before it was put under the radical. This brings him to a standstill when he remembers that the *even root* of a negative quantity is *pronounced impossible.* The seeming conflict, $\sqrt{a^2} = +a$, and $\sqrt{a^2} = -a$, is reconciled when it is known that $\sqrt{-a} \times \sqrt{-a} = \sqrt{a^2}$. The trace of the imaginary quantity is thus preserved.

The safest and simplest direction to the pupil is to put each of the imaginary factors into the form of $\sqrt{-a^2} = a\sqrt{-1}$, and then apply the same rules as in other Radical Quantities.

The following will illustrate this remark:

1. Find the sum of $\sqrt[4]{-1}$ and $\sqrt[4]{-16}$.

Solution. $\sqrt[4]{-16} = 2\sqrt[4]{-1}$;

and

$$\sqrt[4]{-1} + \sqrt[4]{-16} = \sqrt[4]{-1} + 2\sqrt[4]{-1} = 3\sqrt[4]{-1}. \quad \textit{Ans.}$$

2. From $3 - \sqrt{-64}$ take $2 + \sqrt{-1}$.

Solution. $3 - \sqrt{-64} = 3 - 8\sqrt{1-}$;

whence $3 - 8\sqrt{-1} - (2 + \sqrt{-1}) = 1 - 9\sqrt{-1}$. *Ans.*

3. Multiply $2\sqrt[6]{-4}$ by $3\sqrt[6]{-16}$.

Solution. $2\sqrt[6]{-4} = 2\sqrt[6]{4}\,\sqrt[6]{-1}$;

$$3\sqrt[6]{-16} = 3\sqrt[6]{16}\,\sqrt[6]{-1};$$

whence

$$2\sqrt[6]{4}\,\sqrt[6]{-1} \times 3\sqrt[6]{16}\,\sqrt[6]{-1} = 6\sqrt[6]{64}\,\sqrt[6]{-1}$$
$$= 6 \times 2(-1) = -12. \quad \textit{Ans.}$$

4. Divide $14 - \sqrt{15} - (7\sqrt{3} + 2\sqrt{5})\sqrt{-1}$ by $7 - \sqrt{-5}$.

Solution. $14 - \sqrt{15} - (7\sqrt{3} + 2\sqrt{5})\sqrt{-1} \,|\, 7 - \sqrt{-5}$

$$\underline{14 - 2\sqrt{5}\sqrt{-1}} \qquad\qquad 2 - \sqrt{-3}$$
$$-7\sqrt{-3} - \sqrt{15}$$
$$-7\sqrt{-3} - \sqrt{15}.$$

Radical Equations.

In the solution of complex radical equations very much depends upon the judgment and skill of the operator. Solutions are frequently lengthened unnecessarily because the operator does not detect how to contract the work he has to do. No general direction can be given for the solution of all problems under this head. If a problem presents any difficulty, the first thing to be decided is, in what does the difficulty consist, and how can it be avoided? To make this suggestion more forcible, I will solve a problem which presents some difficulty to the average pupil.

Problem.

$$\sqrt{(1+a)^2 + (1-a)x} + \sqrt{(1-a)^2 + (1+a)x} = 2a.$$

It is evident that the equation must be freed from radicals, whence, squaring both members, collecting terms, arranging, and reducing, we have

$$\sqrt{(1-a^2)^2 + 2x(1+3a^2) + (1-a^2)x^2} = (a^2-1) - x.$$

Squaring again, we have

$$(1-a^2)^2+2x(1+3a^2)+(1-a^2)x^2$$
$$=(a^2-1)^2-2(a^2-1)x+x^2.$$

Omitting equal quantities, and dividing by x, we have

$$a^2x^2=8a^2x, \quad \text{and} \quad x=8.$$

The theory of the solution is this: To keep the radical quantities on the same side of the equality; then to throw off the radical sign; collect and arrange terms, until both members are free from radicals.

The learner is at liberty to employ what he has previously learned in solving a problem unless he is restricted to a particular form of solution.

Quadratic Equations.

The learner has now made sufficient progress in Algebra to take hold of equations in a more comprehensive manner than he has hitherto been prepared to do.

If all quadratic equations appeared under the form of *perfect squares* in both members, then it would be unnecessary to teach methods "of completing the square." However, the philosophy underlying this subject should be clearly presented to the pupil or class, and the necessity that existed which caused analysts to search for such a principle, and without the discovery of which no great progress could have been made, ought to be clearly and distinctly emphasized. The typical quadratic equation,

$$Ax^2+Bx+C=0,$$

is not necessarily a perfect power, and unless it is it must be made one by "completing the square," which consists in adding the same quantity to each member of the equation in order to make the left-hand member a perfect power, while the right-hand member may be or may not be a perfect power.

With a class of beginners, it is far better to teach one method of completing *the square* before attempting to teach another. The learner must have within himself the means of testing his own work, and one method thoroughly taught and well understood gives him this test.

Pursuing the same plan as elsewhere indicated, let the pupil begin with the simple exercises first. Then the first step in the solution is to put the equation into one of the four simplest quadratic forms. That is, the coefficient of x^2 is unity. Secondly, to add the square of half the coefficient of x to both members. Thirdly, to extract the square root of both members. Fourthly, to solve the simple equation.

For a while, at least, the pupils should actually complete the square; but further on they should omit that part of the work and write out the values of the unknown quantity from the original equation, except in very complicated

Solution of the General Form.

It is evident that the *General Form*, $x^2 \pm 2px = \pm q$, contains four, and only four, special forms.

Taking the special forms in order, we have

(1) $x^2 + 2px = q$; whence $x = -p \pm \sqrt{q + p^2}$.

(2) $x^2 - 2px = q$; " $x = p \pm \sqrt{q + p^2}$.

(3) $x^2 + 2px = -q$; " $x = -p \pm \sqrt{-q + p^2}$.

(4) $x^2 - 2px = -q$; " $x = p \pm \sqrt{-q + p^2}$.

After specialization follows generalization, and this logical unfoldment is the chart placed in the teacher's hand by which he guides his classes successfully through Algebra.

Since the *general form* of the quadratic embraces *four special forms*, let us examine the roots of these *special*

forms. In (1) and (3) the roots are the same except q under the radical; and (2) and (4) are the same with that exception. The difference between (1) and (2) is in the sign of p not under the radical, and a like difference exists between (3) and (4). In other words, the values are all alike except the algebraic signs of p and q. Consequently the question is narrowed to this: Can all these special forms be brought under one comprehensive statement which will include them all? A trial statement will help to determine the matter. Here it is: For the value of the unknown quantity, write half the coefficient of the first power with its sign changed, plus or minus the square root of the sum of the second member and the square of half the coefficient of the first power of the unknown quantity.

It will be seen that the symbolized values of the unknown quantity are very much more easily written out than expressed in words. The value of the generalization is this, the pupil writes the value of the unknown quantity without taking the time and trouble to complete the square. He throws away his crutches and walks.

As another illustration of the advantage arising from generalization, let us take the following:

$$cx^2 \pm 2px = \pm q.$$

Here again are four special cases:

(1) $cx^2 + 2px = q$; whence $x = \dfrac{-p \pm \sqrt{p^2 + cq}}{c}$.

(2) $cx^2 - 2px = q$; " $x = \dfrac{p \pm \sqrt{p^2 + cq}}{c}$.

(3) $cx^2 + 2px = -q$; " $x = \dfrac{-p \pm \sqrt{p^2 - cq}}{c}$.

(4) $cx^2 - 2px = -q$; " $x = \dfrac{p \pm \sqrt{p^2 - cq}}{c}$.

From the above the pupil should write immediately the values of x, which may be expressed thus: *The unknown quantity is equal to half the coefficient of x with its sign changed,* plus or minus the square root of half the coefficient of x squared and the product of the absolute term by the coefficient of x, and then dividing the whole by the coefficient of x.

Sometimes it is desirable to avoid the occurrence of fractions in solving equations of the second degree. The Hindu Method will enable the operator to write out the values of x without going through the steps of multiplying, reducing, and evaluating.

To solve a problem by this method, I select the following:

Given $$cx^2 + ax = b \text{ to find } x.$$

Solution. Multiplying by $4c$ we have $4c^2x^2 + 4acx = 4bc$. Adding a^2 to both numbers, we have

$$4c^2x^2 + 4acx + a^2 = a^2 + 4bc.$$

Extracting

$$2cx + a = \pm\sqrt{a^2 + 4bc};$$

whence $$x = \frac{-a \pm \sqrt{a^2 + 4bc}}{2c}.$$

Let us take the general form $cx^2 \pm ax = \pm b$, and separate it into the four special forms; we then have

(1) $cx^2 + ax = b$; whence $x = \dfrac{-a \pm \sqrt{a^2 + 4bc}}{2c}$.

(2) $cx^2 - ax = b$; " $x = \dfrac{a \pm \sqrt{a^2 + 4bc}}{2c}$.

(3) $cx^2 + ax = -b$; " $x = \dfrac{-a \pm \sqrt{a^2 - 4bc}}{2c}$.

(4) $cx^2 - ax = -b$; " $x = \dfrac{a \pm \sqrt{a^2 - 4bc}}{2c}$.

Comparing these results with the equations from which they are derived, we are enabled to write the values of x at once from the equation as follows:

The unknown quantity is equal to the coefficient of x with its sign changed, plus or minus the square root of the coefficient of x after it is added to four times the product of x into the known term, and the whole divided by twice the coefficient of x.

The Roots of Quadratic Equations.

It took a long time in the development of the science of Algebra for men to discover any relations existing among the coefficients of the unknown quantity and the roots of that quantity. In an equation of one unknown quantity of whatever degree certain relations always exist, and a knowledge of these conditions helps us to understand the true nature of the equation. For instance, the equation $x^2 + 2ax = -b$ has two roots,

$$x = -a + \sqrt{a^2 - b}, \quad \text{and} \quad x = -a - \sqrt{a^2 - b}.$$

1. The sum of these roots is $-2a$, that is, the coefficient of x with its sign changed.

2. $(-a + \sqrt{a^2 - b}) \times (-a - \sqrt{a^2 - b}) = b$, that is, the product of the roots is equal to the absolute term, or the zero power of x.

The object of this discussion is intended to show the learner that every equation of the second degree containing one unknown quantity is composed of the product of two binomials, and that if we are able to discover these factors by inspection, it would be unnecessary to resolve the equation by the roundabout method of completing the square.

In all treatises on Algebra discussions on the nature and limits of the roots of the four special forms of Quadratic Equations will be found, and which are valuable points to be emphasized in teaching this part of the subject not only

for sharpening the learner's analytical powers in investigating conditions implied in the results, but also for the bearing on the Theory of Equations in general. The pupil, unless properly instructed, gets the idea that the coefficients and absolute term are accidental affairs in the mechanism of equations. He fails to see that we start with roots to make equations rather than that we have equations to find roots. Instead of taking the equation as the essence, it is the developed exponent of the root involved in it, and these are the objects the pupil is trying to find. To solve a problem is the simplest kind of work compared to the discussion of the problem after it is solved. A problem is never thoroughly understood till its nature and limitations are fully comprehended. I will endeavor to put this in a still stronger light. Suppose any equation is reduced to one of the four special forms, then let the pupil or class answer such questions as:

1. What in the sum of the roots?
2. They are equal to what? Illustrate.
3. What is the product of the roots?
4. Equal what?
5. Are the roots real or imaginary?
6. Are the signs alike or unlike? Why?
7. If unlike, what is the greater sign?
8. Under what conditions will the roots be real? Imaginary? When equal? When numerically equal with opposite signs?

Special Artifices.

Mathematicians have resorted to many artifices for the purpose of solving special forms of Quadratic Equations. These artifices consist in transforming or changing the equations into simpler forms which can be more readily reduced than the original ones. It is in this special line of work that algebraic skill shows itself to the best advantage.

Refractory equations bristle, it may be, with difficulties; and unless the operator is able to plan a successful scheme of reduction, he is baffled at every point.

The following will indicate partly the meaning of tentative processes resorted to by algebraists:

Given $x + y = 3$, $xy = p$, to find the value of

$$x^2 + y^2, \quad x^3 + y^3, \quad x^4 + y^4, \quad x^5 + y^5,$$

in terms of s and p.

The results are:

$$\begin{aligned} x^2 + y^2 &= s^2 - 2p; \\ x^3 + y^3 &= s^3 - 3ps; \\ x^4 + y^4 &= s^4 - 4ps^2 + 2p^2; \\ x^5 + y^5 &= s^5 - 5ps^3 + 5p^2s. \end{aligned}$$

In the solution of such problems as the following these equivalents may be employed advantageously:

$$\left\{ \begin{aligned} x + y &= 4 \\ x^4 + y^4 &= 82 \end{aligned} \right\}; \qquad \left\{ \begin{aligned} x + y &= 11 \\ x^3 + y^3 &= 407 \end{aligned} \right\}.$$

But such substitutions are not absolutely necessary, since these and other similar problems can be readily solved by combining the equations as they are given.

When two quadratic equations are symmetrical with respect to the two unknown quantities, they may frequently be solved by substituting for the *sum* and *difference* of the two other quantities.

Thus:

$$\begin{aligned} x + y &= a, \\ x^5 + y^5 &= b, \text{ to find } x \text{ and } y. \end{aligned}$$

Here put $x = s + d$, $y = s - d$; and $s = \dfrac{a}{2}$.

Solution.

$$\begin{aligned} x^5 &= s^5 + 5s^4d + 10s^3d^2 + 10s^2d^3 + 5sd^4 + d^5, \\ y^5 &= s^5 - 5s^4d + 10s^3d^2 - 10s^2d^3 + 5sd^4 - d^5, \end{aligned}$$

and $x^5 + y^5 = 2s^5 + 20s^3d^2 + 10sd^4 = b;$

or $\quad x^5 + y^5 = d^4 + \frac{a^2d^2}{2} = \frac{16b - a^5}{80a}.$

Completing the square and reducing, the value of d can be found, and then the values of x and y.

But these equations are also easily solved in the following manner:

$$\begin{array}{ll}
(1) & x + y = a; \\
(2) & x^5 + y^5 = b; \\
(3) = (2) \div (1) & = x^4 - x^3y + x^2y^2 - xy^3 + y^4 = \frac{b^5}{a} \\
 & = x^4 + y^4 - xy(x^2 + y^2) + x^2y^2 = \frac{b^5}{a}; \\
(4) = (1)^2 & = x^2 + y^2 = a^2 - 2xy; \\
(5) = (4)^2 & = x^4 + 2x^2y^2 + y^4 = a^7 - 4a^2xy + 4x^2y^2 \\
 & = x^4 + y^4 = a^4 - 4a^2xy + 2x^2y^2.
\end{array}$$

Hence

$$\begin{array}{ll}
(6) = (3) & = a^4 - 4a^2xy + 2x^2y^2 - xy(a^2 - 2xy) + \\
 & \qquad x^2y^2 = \frac{b^5}{a}; \\
(7) = (6) & = 5x^2y^2 - 5a^2xy = \frac{b^5}{a} - a^4.
\end{array}$$

Whence $\qquad xy = \dfrac{a^2 \pm \sqrt{\dfrac{4b^5 + a^5}{5a}}}{2}.$

By combining with (1), the values of x and y are found.

Given (1) $x^2 + y^2 = 13$, and (2) $x^3 + y^3 = 35$, to find x and y.

Put $\qquad x + y = s, \qquad$ and $\qquad xy = p.$

Solution.

$$\begin{array}{ll}
(3) = (1) & = s^2 - 2p = 13; \\
(4) = (2) & = s^3 - 3ps = 35; \\
(5) = (4) \times 2 & = 2s^3 - 6ps = 70;
\end{array}$$

$$(6) = (3) \times 3s \quad = 3s^3 - 6ps = 39s;$$
$$(7) = (6) - (5) \quad = \quad s^3 - 39s = -70;$$
$$(8) = (7) \times s \quad = \quad s^4 - 39s^2 - 70s;$$
$$(9) = (8) + 25s^2 = \quad s^4 - 14s^2 = 25s^2 - 70s;$$
$$(10) = (9) + 49 \quad = \quad s^4 - 14s^2 + 49 = 25s^2 - 70s + 49;$$
$$(11) = \sqrt{(10)} \quad = \quad s^2 - 7 = \pm(5s - 7),$$
$$\text{and } s = 5, 2, \text{ or } -7.$$

x and y are found from $x + y = 5$ and $xy = 6$.

Homogeneous equations of the fourth degree are easily solved by making a simple substitution, as is illustrated in the following:

Given (1) $3x^2 + xy = 68$, (2) $4y^2 + 3xy = 160$, to find x and y.

Put $y = nx$.

Solution.

$$(3) = (1) = 3x^2 + nx^2 = 68;$$
$$(4) = (2) = 4n^2x^2 + 3nx^2 = 160;$$
$$(5) \qquad x^2 = \frac{68}{3 + n} = \frac{160}{4n^2 + 3n}.$$

Clearing and reducing, $n = \frac{5}{4}$, and $x = \pm 4$, $y = \pm 5$. The other values of x and y can be readily found.

Very frequently the shortest cut in the solution of a problem is to look at it for several minutes for the purpose of *seeing it well.*

The following is one of this character:

Given (1) $(x^2 + 1)y = xy + 126$,

and (2) $(x^2 + 1)y = x^2y^2 - 744$, to find x and y.

$(3) = (1) \div (2)$ $\quad x^2y^2 - 744 = xy + 126$,

and $\quad xy = 30, \text{ or } -29$,

and $\quad x = 5, \frac{1}{5}, \text{ or } \dfrac{-97 \pm \sqrt{6045}}{58};$

$$y = 6, 150, \text{ or } \frac{1682}{97 \mp \sqrt{6045}}.$$

I will now select a few miscellaneous problems to illustrate some of the methods adopted in solving complicated equations.

Find x, y, and z from the equations:

$$(1) \quad x+y+z=27;$$
$$(2) \quad x^2+y^2+z^2=269;$$
$$(3) \quad x^3+y^3+z^3=2853.$$

Solution.

$(4)=(1) \quad z=27-x-y.$

Putting this value in (2) and (3), we have

$(5)=(2) \quad x^2+xy+y^2-27x-27y=-230;$

$(6)=(3) \quad x^2y+xy^2-27x^2-54xy-27y^2+729x+729y$
$=5610.$

Adding 27 times (5) to (6),

$(7) \quad x^2y+xy^2-27xy=-600;$

$(8)=(5) \quad y^2=27x+27y-xy-x^2-230.$

Substituting in (7),

$(9) \quad x^3-27x^2+230x=600.$

$(10)=(9)\times 4x$, and adding $9x^2-2700x+2500$ to each member, we have

$(11) \quad (2x^2-27x)^2+100(2x^2-27x)+2500=$
$9x^2-300x+2500;$

$(12)=\sqrt{11} \quad 2x^2-27x+50=\pm(3x-50).$

Taking the lower sign, $x=12.$

From (7), $\quad y^2-15y=-50, \quad y=10.$

From (1), $\quad z=5.$

Find x and y from the equations

$(1) \quad (x^2-xy+y^2)(x^2+xy+y^2)=336;$

$(2) \quad (x^4-x^3y+x^2y^2-xy^3+y^4)(x^4+x^3y+x^2y^2+xy^3+y^4)=$
$87296.$

Solution.

$(3)=(1) \quad x^4+x^2y^2+y^4=336;$

$(4)=(2) \quad x^8+x^6y^2+x^4y^4+x^2y^6+y^8=87296.$

Put $\quad x^2+y^2=s, \quad x^2y^2=p,$

and

(5) = (3) $s^2 - p = 336;$

(6) = (4) $s^4 - 3ps^2 + p^2 = 87296.$

Substituting the value of p from (5), and reducing, we have

(7) = (6) $s^4 - 336s^2 = 25600,$

and $s = 20 = x^2 + y^2.$

Putting this value in (5),

$$p = 64 = x^2y^2, \quad \text{or} \quad xy = 8,$$

whence $x = 4, \quad y = 2.$

Find x from the equation

$$(1) \quad 3x^9 - 4x^5 + 6x^4 - 4x = 12.$$

Multiplying the equation by x, it becomes

(2) $3x^{10} - 4x^6 + 6x^5 - 4x^2 = 12x;$

(3) = (2) $4x^2 + 12x + 4x^6 = 3x^{10} + 6x^5.$

Adding $6x^5 + 9$ to each member, and

(4) $4x^2 + 12x + 9 + 4x^6 + 6x^5 = 3x^{10} + 12x^5 + 9;$

or

(5) = (4) $(2x + 3)^2 + 2x^5(2x + 3) = 3x^{10} + 12x^5 + 9.$

Adding x^{10} to each side, and we have

(6) $(2x + 3)^2 + 2x^5(2x + 3) + x^{10} = 4x^{10} + 12x^5 + 9;$

(7) = $\sqrt{(6)}$ $(2x + 3) + x^5 = \pm(2x^5 + 3);$

whence $x = \sqrt[4]{2}.$

Find x and y from the equations

(1) $xy + (x^2 + y^2)(1 + xy + x^2y^2 + x^3y + xy^3) = 87;$

(2) $xy(x^2 + y^2)(x^2 + xy + y^2)(x^2 + y^2 + xy + xy^3 + x^3y) = 1190.$

Solution.

(3) = (1) $(xy + x^2 + y^2) + xy(x^2 + y^2) + xy$
$(x^2 + y^2)(xy + x^2 + y^2) = 87;$

(4) = (2) $xy(x^2 + y^2)(xy + x^2 + y^2)^2 +$
$x^2y^2(x^2 + y^2)^2(xy + x^2 + y^2) = 1190.$

Put $v = xy + x^2 + y^2$, and $w = xy(x^2 + y^2).$

Then

(5) = (3) $v + w + vw = 87,$

and

(6) = (4) $vw(v + w) = 1190.$

From (5) and (6), $v = 7, \qquad w = 10.$

Also we have

(8) $x^2 + y^2 + xy = 7,$

and

(9) $xy(x^2 + y^2) = 10,$

whence $x = 2, \qquad y = 1.$

Find the value of x in the equation

(1) $x^4 - 2ax^3 - 2abx + b^3 = 0.$

Adding $(a^2 + 2b)x^2$ to each member, we have

(2) $$x^4 - 2ax^3 + (a^2 + 2b)x^2 - 2ab + b^3 = (a^2 + 2b)x^2.$$

(3) = $\sqrt{(2)}$, $x^2 - ax + b = \pm x\sqrt{a^2 + 2b},$

and $x^2 - [a \pm (\sqrt{a^2 + 2b})x] = -b;$

whence

$$x = \tfrac{1}{2}\{a \pm \sqrt{(a^2 + b^2)} \pm \sqrt{(2a^2 - 2b \pm 2a\sqrt{(a^2 + 2b)})}\}.$$

The effect upon my mind when I first examined such problems was not very encouraging. I could not see how there could be any method running through the solutions, and as much as I could make out of the different operations was a series of "extraordinary guesses." But, commencing work first with simple problems, I discovered method in *the guesses*, and I now proceed to unfold it by using easy illustrations.

1. Given $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$, to find x.

We can write this equation as follows:

(2) $x^4 - 12x^3 + 36x^2 + 11x^2 - 72x + 36 = 0;$

or

(3) = (2), $(x^2 - 6x)^2 + \qquad 11x^2 - 72x + 36 = 0.$

It is evident that (3) is not yet a perfect square, but if we add x^2 to each member we have

(4) $$(x^2 - 6x)^2 + 12(x^2 - 6x) + 36 = x^2.$$

That is, the $(x^2 - 6x)$ puts (4) into the regular quadratic form, and extracting the square root we have

$$x^2 - 6x + 6 = \pm x;$$

whence $$x = 1,\ 2,\ 3,\ \text{or}\ 6.$$

2. Given $x^4 - 2x^3 + x = 30$, to find x.

Writing it under the following form we have

$$x^4 - 2x^3 + x^2 - x^2 + x = 30;$$

or

$$(x^2 - x)^2 - (x^2 - x) \quad = 30.$$

Adding $\frac{1}{4}$ to each member and the square root taken, the values of x are

$$3,\ -2,\ \frac{3 \pm \sqrt{19}}{2}.$$

Many biquadratic equations can be solved by adding a binomial square to each member.

3. Given $x^4 + 3x^3 + x^2 - 3x = 2$, to find x.

Adding x^2 to each member, we have

(1) $$x^4 + 3x^3 + x^2 + x^2 - 3x = x^2 + 2;$$

or

(2) = (1), $$\left(x^2 + \frac{3}{2}x\right)^2 - \left(\frac{x^2}{4} + 3x\right) = x^2 + 2;$$

(3) = (2) $- \frac{3x^2}{4}$, $$\left(x^2 + \frac{3}{2}x\right)^2 - \left(x^2 + \frac{3x}{2}\right) = \frac{x^2}{4} + \frac{3x}{2} + 2.$$

Adding $\frac{1}{4}$ to (3), we have

(4) $$\left(x^2 + \frac{3}{2}x\right)^2 - \left(x^2 + \frac{3x}{2}\right) + \tfrac{1}{4} = x^2 + \frac{3x}{2} + \frac{9}{4}.$$

(5) = $\sqrt{(4)}$, $$x^2 + \tfrac{3}{2}x - \tfrac{1}{2} = \pm(x + \tfrac{3}{2});$$

whence $$x = 1,\ -1,\ -1,\ \text{or}\ -2.$$

4. Find x from the equation

(1) $$x^4 - 7x^3 + 9x^2 + 27x = 54.$$

$(2) = (1) \times 4 + 25x^2,\ 4x^4 - 28x^3 + 49x^2 + 12x^2 + 108x$
$= 25x^2 + 216.$

Subtracting $150x$ from each member, (2) becomes

(3) $(2x^2 - 7x)^2 + 6(2x^2 - 7x)$
$= 25x^2 - 150x + 216;$

$(4) = (3) + 9,\ (2x^2 - 7x)^2 + 6(2x^2 - 7x) + 9$
$= 25x^2 - 150x + 225;$

$(5) = \sqrt{(4)},\quad 2x^2 - 7x + 3 = \pm(5x - 15);$

whence $x = 3,\ 3,\ 3,$ or $-2.$

5. Find x from the cubic equation

(1) $x^3 - 8x^2 + 19x - 12 = 0;$

$(2) = (1) \times x,\ x^4 - 8x^3 + 19x^2 - 12x = 0;$

$(3) = (2),\quad (x^2 - 4x)^2 + 3(x^2 - 4x) = 0;$

$(4) = (3) + \frac{9}{4},\ (x^2 - 4x)^2 + 3(x^2 - 4x) + \frac{9}{4} = \frac{9}{4};$

$(5) = \sqrt{(4)},\quad x^2 - 4x = -\frac{3}{2} \pm \frac{3}{2};$

whence $x = 4,\ x = 3,\ x = 1.$

6. Find x from the equation

(1) $x^4 - 27x^2 + 14x + 120 = 0.$

$(2) = (1),\quad x^4 - 26x^2 + 169 = x^2 - 14x + 49;$

$(3) = \sqrt{(2)},\ x^2 - 13 = \pm(x - 7);$

whence $x = 3,\ 4,\ -2,$ or $-5.$

From the six preceding solutions the learner will see that problems can be solved without much difficulty, and that proficiency is obtained by intelligent practice. Exercises may be selected from the problems in the various treatises on Algebra, and with a month or more of practice on such problems, after the completion of ordinary chapters on Quadratics, the learner will be able to solve many problems which would baffle him except by the ordinary theory of equations. Of course there are methods of solving numerical equations of the third and fourth degrees; but the art of solving them by quadratic methods is one of the most

valuable disciplines in the entire scope of mathematical science.

Let the exercises be simple at first, gradually growing more difficult—not so difficult as to cause discouragement, and both teacher and learner or class will soon be astonished at the progress made in a very short time.

After completing any ordinary treatise on Algebra, I spent some time in having my classes solve such problems selected from various sources. From a knowledge of what average classes can do in this kind of work, I do not hesitate to recommend it unqualifiedly.

Ratio and Proportion.

A clear and precise notion of *all definitions* is the first thing to be mastered under this head. A definition should never be the least bit hazy in the mind. Its meaning should always be strong and pronounced. Make clean work then among the definitions. Every one should be thoroughly comprehended. Confusion of terms here denotes indistinctness of thought.

The second is the demonstration of all propositions. Of course the learner when studying Arithmetic got a sort of running idea of Ratio and Proportion, and some of that knowledge he still has, and the danger now to be guarded against is his inclination to bring all propositions and problems under one general proposition—the product of the *extremes* is equal to the product of the *means*—and then solve by algebraic equations. However, in the demonstration of the propositions, the fixing process is best accomplished by using numbers substituted for the quantities. Especially is this true of those who experience difficulty in imagining quantities to be numbers without making the actual substitution. The tendency of the mind undoubtedly is to reduce every new acquisition to a close connection with something previously known, and here the imaginary

process is still carried on, and in some cases, at least, with great difficulty.

Another precaution is necessary, and it is—

In proportion, look to it that all problems are solved by proportion. Unless this suggestion is strictly enforced, the learner or class may never see the beauty and necessity for doing so. When working in proportion, the problems must be solved according to the propositions of proportion. The most beautiful solutions we have in Algebra are those effected by an application of the principles involved in the theory of proportion.

To illustrate my meaning, the following equations are selected:

(1) Given $xy = 24$,

(2) $x^3 - y^3 : (x - y)^3 :: 19 : 1$, to find x and y.

$$x^2 + xy + y^2 : x^2 - 2xy + y^2 :: 19 : 1. \quad \text{What Prop.?}$$

$$3xy : x^2 + xy + y^2 :: 18 : 19. \quad \text{What Prop.?}$$

$$xy : x^2 + xy + y^2 :: 6 : 19. \quad \text{What Prop.?}$$

$$xy : (x + y)^2 :: 6 : 25. \quad \text{What Props.?}$$

$$4xy : (x + y)^2 :: 24 : 25. \quad \text{What Props.?}$$

$$(x + y)^2 : (x - y)^2 :: 25 : 1. \quad \text{What Props.?}$$

$$x + y : x - y :: 5 : 1. \quad \text{What Prop.?}$$

$$2x : 2y :: 6 : 4. \quad \text{What Prop.?}$$

$$x : y :: 3 : 2. \quad \text{What Prop.?}$$

$$y = \frac{2x}{3}. \quad \text{What Prop.?}$$

By (1), $xy = 24$; whence $x = \pm 6$; $y = \pm 4$.

Teacher, require at each step the reason therefor, if you wish your classes to do the work properly. Proportion is so frequently used in other branches of mathematics that its importance can hardly be estimated from its limited treatment in Algebra. The groundwork, however, needs to be well laid in Algebra so far as our books teach it, and then continued and applied in the more advanced branches.

Series.

The subject of series is usually passed over rather lightly by most teachers of mathematics, and more particularly by those who do not understand what an important part Series play in many questions occurring in other branches of investigation. As a preparation for entering fully upon the subject, the *first* thing is to get a good definition of "a series."

Second. The Law of the Series.

Third. The Terms.....
$$\begin{cases} a = \text{First Term;} \\ l = \text{Last Term;} \\ d = \text{Common Difference;} \\ r = \text{Ratio;} \\ n = \text{Number of Terms;} \\ s = \text{Sum of the Series.} \end{cases}$$

Fourth. The Kind of the Series.
Fifth. Finite or Infinite.
Sixth. Converging or Diverging.
Seventh. Increasing or Decreasing by Differences or by Ratios.

The most important special cases are those of arithmetical and geometrical series. With the five different terms having any three given, the other two can be found. That is, each of these two progressions will give twenty formulas. For arithmetical progression a valuable exercise for the learner is to derive all the other formulas from the equations $l = a + (n-1)d$, and $S = (a + l)\frac{n}{2}$.

As soon as he derives a formula he should substitute numerical values for the letters in the formula. In this manner each formula should be found, and some of the principal ones ought to be remembered, which does away with the too common practice of referring to the book for everything.

Geometrical Progression.

This series, in an algebraic point of view, affords much greater scope for a display of skill than that of Arithmetical Progression. The formulas are more complicated, and four of them involve logarithms with which the learner is not yet supposed to be acquainted. However, the other sixteen formulas can be deduced from the equations:

$$l = ar^{n-1} \text{ (1), and } S = \frac{lr - a}{r - 1}, \text{ or } \frac{a - lr}{1 - r} \text{ (2).}$$

In these equations a, r, n are known quantities. Again, the learner should work out the formulas and make the substitutions in the manner described under Arithmetical Progression.

Another good exercise is to have the learner or class point out the corresponding or correlative formulas in these two progressions.

Two problems are given to illustrate, in part, the nature of the algebraic processes required to solve such questions.

1. The sum of five numbers in Arithmetical Progression is 25; their continued product is 945. Find the numbers.

Solution. Let $(x - 2y)$, $(x - y)$, x, $(x + y)$, $x + 2y$, be the numbers. Then

(1) $(x - 2y) + (x - y) + x + (x + y) + (x + 2y) = 25;$

or, $x = 5.$

(2) $(5 - 2y)(5 - y)\,5(5 + y)(5 + 2y) = 945;$

or $4y^4 - 125y^2 = -436;$

or $y^2 = 4,\ y = \pm 2.$

Hence the numbers are 1, 3, 5, 7, 9. *Ans.*

2. The sum of six numbers in geometrical progression is 1365, and the sum of the extremes is 1025. Find the numbers.

Solution. Let the numbers be x, xy, xy^2, xy^3, xy^4, xy^5.

Then, by the nature of the series and the conditions of the problem, we have

(1) $$\frac{xy^6 - x}{y - 1} = 1365;$$

(2) $$x + xy^5 = 1025 = (y^5 + 1)x = 1025.$$

Equating the values of x in (1) and (3), we have

(3) $$\frac{1365(y-1)}{y^6 - 1} = \frac{1025}{y^5 + 1};$$

(4) = (3), $$\frac{273}{y^4 + y^2 + 1} = \frac{205}{y^4 - y^3 + y^2 - y + 1};$$

or $$68\left(y^2 + \frac{1}{y^2}\right) - 273\left(y + \frac{1}{y}\right) = -68;$$

or $$68\left(y + \frac{1}{y}\right)^2 - 273\left(y + \frac{1}{y}\right) = 68.$$

Reducing, $$y + \frac{1}{y} = \frac{17}{4}.$$

Completing the reduction, the numbers are 1, 4, 16, 64, 256, 1024.

The following problem and solution are inserted because they lie beyond the usual list of problems in university and college algebras :

3. The sum of seven numbers in geometrical progression is 127, and the sum of their squares is 5461. Find the numbers.

Solution. Let x, xy, xy^2, xy^3, xy^4, xy^5, xy^6 be the numbers. Then

(1) $$x + xy + xy^2 + xy^3 + xy^4 + xy^5 + xy^6 = 127;$$

(2) $$x^2 + x^2y^2 + x^2y^4 + x^2y^6 + x^2y^8 + x^2y^{10} + x^2y^{12} = 5461.$$

By formula,

(3) $$\frac{xy^7 - x}{y - 1} = 127;$$

(4) $$\frac{x^2y^4 - x^2}{y^2 - 1} = 5461;$$

(5) = (4) ÷ (3), $$\left(\frac{y^7+1}{y+1}\right)\left(\frac{y-1}{y^7-1}\right) = \frac{43}{127};$$

(6) = (5), or $$\frac{y^6 - y^5 + y^4 - y^3 + y^2 - y + 1}{y^6 + y^5 + y^4 + y^3 + y^2 + y + 1} = \frac{43}{127}.$$

Clearing and arranging, we have

(7) $$84y^6 - 170y^5 + 84y^4 - 170y^3 + 84y^2 - 170y + 84 = 0.$$

This is a *recurring* equation.

(8) = (7) ÷ y^3, $$84y^3 - 170y^2 + 84y - 170 + \frac{84}{y} - \frac{170}{y^2} + \frac{84}{y^3} = 0.$$

Put $\left(y + \frac{1}{y}\right) = s$. Then substituting,

(9) = (8), $$42s^3 - 85s^2 - 84s = -85;$$

whence $s = \frac{5}{2}$, $y = 2$, $x = 1$.

The numbers are 1, 2, 4, 8, 16, 32, 64.

The learner should familiarize himself with the various artifices employed by algebraists for representing series. No very definite rule can be laid down for the solution of all questions; but problems may be grouped, and the method of solution for that group ascertained, and so on for other groups. Here, however, as in all other cases, success depends upon the ingenuity of the operator whose skill enables him to do the work in the easiest and simplest manner. Geometrical progression affords a fine field for the algebraist to exercise all his powers.

Harmonical Progression.

This progression, which is usually lightly touched upon in elementary treatises, derives its importance primarily from its connection with musical sounds. That is, if a series of strings of the same substance, but whose lengths are proportional to the numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$,

$\frac{1}{6}$, $\frac{1}{7}$, etc., and these strings be stretched with equal force or weights, and any two be sounded together, harmony, as it is called, is produced.

Any three quantities as a, b, c, are in Harmonical Progression when $a : c :: a - b : b - c$.

From the nature of this series two definitions have been formulated as follows :

1. Three quantities are in Harmonical Progression when the first is to the third as the difference of the first and second is to the difference of the second and third.

2. Three quantities are in Harmonical Progression when their reciprocals are in Arithmetical Progression.

The first definition is better expressed by the proportion itself than when converted into words, while the second definition is really a proposition susceptible of proof.

Also Harmonic Ratio and Proportion branch off into modern Geometry. This extension, in connection with its relation to Music, is sufficient apology for reference to it in this place.

Continuation of Series.

There is no limit to the different kinds of series, and a very large treatise would not exhaust the subject. It frequently happens that in the solution of a problem approximate results are all that can be obtained, and recourse must be had to some known series to even approximate the value of the unknown quantity. The law of a series must always express two definite facts :

1. The rate of the increase or decrease of the series;

2. The intervals of time at which its values are taken for the terms of the series.

The learner should learn as much of the nature of series as is usually found in our best American text-books on this subject.

After the general discussion following each special series,

if the learner attempts to generalize the problems, he will find that they can be thrown into four principal groups:

1. **To find any required term of a series.**
2. **To interpolate a term or terms of a series.**
3. **To find the sum of a series.**
4. **To revert a series.**

To get at the way a series is formed we may regard it as resulting from dividing the numerator of a fraction by its denominator, or from involution or evolution. Algebraic quantities are expanded into series in four ways:

1. **By the Method of Division.**
2. **By the Method of Undetermined Coefficients.**
3. **By the Method of Involution.**
4. **By the Method of Evolution.**

The method of expansion by Undetermined Coefficients is one with which the learner ought to be very familiar. It also affords a good exercise in determining the coefficients in the series. In decomposing rational fractions it is also advantageously employed, and it is often used for this purpose in the Calculus.

The third and fourth methods are so closely connected with the Binomial Theorem that they naturally fall under it; but the Multinomial Theorem for expanding polynomials is regarded as less complicated than the Binomial Theorem.

Reversion of Series.

Reversion of Series is so frequently used in the solution of equations that all algebraists recognize its importance. The value of this process depends upon its application in the more advanced mathematics. Many elementary things must be learned in the lower branches ready for use in the higher ones; that is, if the learner intends to study beyond the merest rudiments.

The Differential Method.

This method has for its object the summation of a series by ascertaining the successive differences of its terms. It is easily developed into a series. Then the close connection between it and the Binomial Formula will be seen.

The questions that claim most attention in the treatment of this series are: **To find any term of a series by first deducing the formula; to find the sum of any number of terms of a series by first deducing the formula; the interpolation of terms in a series of equidistant terms; and the application to piling balls.**

A few problems will now be solved illustrating several of the preceding series.

1. Find the nth term of an harmonical progression.

Solution. Let L be the last term; then we have

$$L = \frac{1}{a} + (n-1)\left(\frac{a-b}{ab}\right) = \frac{a(n-1) - b(n-2)}{ab}.$$

Inverting,

$$\frac{ab}{a(n-1) - b(n-2)}. \quad \textit{Ans.}$$

2. Expand 1 into a series.

Solution.

$$1. \quad \frac{x}{1-1+x} = x + x + x + x + x, \text{ etc.}$$

$$2. \quad \frac{x}{x+1-1} = x - x^2 + x^3 - x^4, \text{ etc.}$$

$$3. \quad \frac{x}{x+1-1} = 1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3}, \text{ etc.}$$

3. Decompose

$$\frac{13 + 21x + 2x^2}{1 - 5x^2 + 4x^4}.$$

Solution. Let

$$13 + 21x + 2x^2 = A(1-x)(1-4x^2) + B(1+x)(1-4x^2) + C(1-x^2)(1-2x) + D(1-x^2)(1+2x).$$

Now assume

$$x = -1, \quad x = 1, \quad x = -\tfrac{1}{2}, \quad x = \tfrac{1}{2},$$

and we have

$$\frac{1}{1+x} - \frac{6}{1-x} + \frac{2}{1+2x} + \frac{16}{1-2x}. \quad \textit{Ans.}$$

4. Find the sum of n terms of the series $1(m+1)$, $2(m+2)$, $3(m+3)$, $4(m+4)$, $5(m+5)$, etc.

Solution. By the order of differences, $D' = (m+3)$, $D^2 = 2$, $D^3 = 0$.

$$\therefore \quad S = n(m+1) + \frac{n(n-1)}{2}(m+3) + \frac{n(n-1)(n-2)}{2 \times 3} \times 2 = \frac{n(n+1)(1+2n+3m)}{1 \times 2 \times 3}.$$

Binomial Theorem.

The learner already knows how to expand a binomial when the exponent is a positive integer. He should now be held to a rigorous demonstration of this beautiful theorem.

The formula ought to be as well known to the pupil as the simplest theorem in the book. When he has committed the formula after first demonstrating it, then he can make all necessary substitutions for the exponent whatever its character. In the same expansion, let n be assumed as integral and positive; integral and negative; fractional and positive; fractional and negative. By whatever method the theorem is demonstrated, the fact should never be lost sight of, that the learner or class should always remember

the theorem itself. Some things are to be learned and remembered. This is one of those truths. Some sort of demonstration of it is found in all books. Newton, it is claimed, never demonstrated it.

The one fact, $(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \, . \, 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \, . \, 2 \, . \, 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1 \, . \, 2 \, . \, 3 \, . \, 4}a^{n-4}b^4 +$, etc., should be at the finger's end, ready for use whenever needed.

Logarithms.

Great clearness is necessary in the outset in teaching Logarithms. The definition is often not comprehended by a class of pupils, and no doubt many use a Table of Logarithms without having the remotest conception of what a logarithm is. In my opinion the obscurity comes from the lack of a sharp distinction between the *measure of the terms* of a *quantity,* and the *measure of its factors.* The measure of the terms of a number is the measure of its effect when added or subtracted, while the measure of its factors is found by multiplication or division, i.e., $12 \times 3 = 36$; or $12 \div 4 = 3$. Or a more obvious distinction is, that the measure of terms is effected by a *coefficient,* and the measure of the factors by an *exponent.* To contract computation in which numbers are used as factors, we have a Table of Logarithms. That is, the exponents are the logarithms.

Those points requiring special attention are :

1. **The Definition of Logarithms.**
2. **The Base of the System.**
3. **Meaning of the equation $a^x = n$. What is a? x? n?**
4. **Why cannot 1 or -1 be used as the Base of a System?**
5. **The difference between Characteristic and Mantissa.**
6. In any system the logarithm of the base is 1. Why?

7. In any system whose base is greater than 1, the logarithm of 0 is $+\infty$. Why?

8. In a system whose base is positive, a negative quantity has no real logarithm. Why?

9. The logarithm of a product is equal to the sum of the logarithms of its factors. Demonstrate.

10. The logarithm of a quotient is equal to the remainder obtained by subtracting the logarithm of the divisor from that of the dividend. Demonstrate.

11. The logarithm of any power of a number is equal to the product of the exponent of the power and the logarithm of the number. Demonstrate.

12. The logarithm of any root of a number is equal to the quotient obtained by dividing the logarithm of the number by the index of the root. Demonstrate.

13. How to find logarithms of numbers in a Table of Logarithms between given limits.

14. What the characteristic must be? When negative?

15. How to use such a Table?

16. What is meant by "Arithmetical Complement"?

17. How to change "our common system" to the Napierian system. The converse.

18. Solution of numerical problems by Logarithms.

19. The Solution of Exponential Equations.

20. Application of Logarithms to the solution of Compound Interest and Annuity Problems.

21. Discussion of Exponential and Logarithmic Series.

22. To calculate Logarithms.

Permutations and Combinations.

Of late years a tremendous impulse has been given to these two subjects in connection with that of the Theory of Probabilities.

As much as can be done in this connection is to direct the pupil how to acquire a knowledge of this important

department of Modern Mathematics. The problems range all the way from Arithmetic through the Calculus. But in teaching the subject to beginners, it is preferable to begin with very simple arithmetical questions. By degrees lead up then to the algebraic problems.

The teacher can illustrate the fundamental principles of Permutation and Combination by using either figures or letters. Teach Permutation only, till its nature is firmly fixed in the mind. Use two or three letters first, then four, and so on. Follow this by Combinations with the same objects. Then compare the two, and let the differences be noted. As an illustration of the above suggestion, suppose it be required to find the number of permutations that can be formed of the letters *a, b, c,* taken three in a set. The total number of permutations is equal to the total number of possible arrangements. The sets are: *abc, acb, bac, bca, cab, cba.* The combinations without repetition are: *abc, acb, bac.* This distinction is fundamental.

$$\text{Permutations}: \begin{cases} abc \\ acb \\ bac \\ bca \\ cab \\ cba \end{cases} \qquad \text{Combinations}: \begin{cases} abc \\ acb \\ bac \end{cases}$$

Having given a sufficient number of exercises of this character, the first problems following will be included under this head: "To find the number of permutations of n things taken 1, 2, 3, 4, ... m in a set, "$n > m$."

Suppose n equals 4. Then if we have $a, b, c, d,$ and one letter at a time be taken, there are

	4	permutations	$= a + b + c + d =$	4
Two letters	4×3	"	=	12
Three "	$4 \times 3 \times 2$	"	=	24
Four "	$4 \times 3 \times 2 \times 1$	"	=	24
			Total permutations,	64

The permutations of the letters *a, b* are — *ab, ba*; the combination is *ab*.

If *a, b, c* be permuted in sets of three, we have *abc, acb, bac, bca, cab, cba*; but the total number of combinations are *abc, acb, bac.*

Permutation is placing a number of things in all orders possible, while combination is the number of different collections that may be made of a number of things, so that no two collections shall be the same.

The number of permutations of which n letters are susceptible is equal to the product of the natural numbers from 1 to n. Suppose n equals 3; then *a, b, c,* when permuted, will be

$$n(n-1)(n-2) = 3 \times 2 \times 1 = 6.$$

If we take three letters, as *a, b, c,* we can arrange them in sets of *one, two, three,* thus:

Of one, *a, b, c.*

Of two, *ab, ac*; *ba, bc*; *ca, cb.*

Of three, *abc, acb*; *bac, bca*; *cab, cba.*

The number of permutations singly equals n.

The number of permutations in sets of two equals $n(n-1)$.

The number of permutations in sets of three equals $n(n-1)(n-2)$.

The law of extension is obvious.

To find the law of combination, proceed thus: The number of combinations of n things taken singly equals n.

The number of permutations of n letters taken two at a time is $n(n-1)$. Since each combination admits of (1×2) permutations, there are (1×2) times as many permutations as combinations; that is, the combinations $\frac{n(n-1)}{1 \times 2}$.

If n letters or things be taken three at a time, the number of combinations will be $\frac{n(n-\ )(n-2)}{1 \times 2 \times 3}$, and so on.

A great variety of interesting problems can be selected by the teacher and given to the class. Pupils are interested in all such questions because they can see a practical application of the principles underlying each operation. My own experience, too, is that pupils should master Permutation and Combination quite thoroughly before commencing the Theory of Probabilities. The exercises in our more modern treatises on Algebra are sufficient at least to give a class a fair start, and no teacher of Algebra can afford to be ignorant of these subjects. They constitute a bright spot in the algebraic region.

Probabilities.

Under this heading definitions are important; thus:

1. What is the *probability* that an event will happen?
2. What is the improbability?
3. What is a *simple* probability?
4. What is a *compound* probability?
5. What is a *favorable* case?
6. What is an *unfavorable* case?
7. When is an event *dependent?*
8. When is an event *independent?*

Principles.

1. The probability that an event will happen is equal to the number of favorable chances divided by the whole number of chances.

Algebraically thus: $\frac{a}{a+b}$, where a denotes the *favorable* chances and b the *unfavorable* ones. If a equals b, then $\frac{a}{a+b}=\frac{1}{2}$, as in the case of tossing up a coin.

2. That an event will not happen is denoted thus:

$$\frac{b}{a+b}.$$

3. That an event did or did not happen is denoted thus:

$$\frac{a}{a+b}+\frac{b}{a+b}=1=\text{certainty.}$$

4. That of several events of which only one can happen, the chance that some one of them will happen is the sum of all the chances.

5. That two independent events will both happen is the product of their chances of happening.

I sincerely hope all teachers of algebra will make it a point to cultivate an acquaintance with the elementary principles of this beautiful science. It is indeed one of the most attractive fields now before mathematicians.

There is much upon this subject in our newest and freshest text-books, and a great deal more scattered through our mathematical literature. It affords me great pleasure to refer to the many able and elegant solutions found in foreign and native periodicals upon this subject, contributed by the lamented Prof. E. B. Seitz, and by Dr. Artemus Martin, now of Washington City. These two distinguished mathematicians deserve great credit for their researches in this department. Dr. Martin has done, and is still doing, more to popularize sound mathematical scholarship than any other person on this continent.

The only direction that I would give for teaching this subject is, to create an interest and then give the class problems, beginning at first with very simple exercises.

General Theory of Equations.

So far methods of solving equations of the first, second, third, and fourth degrees have been discussed. But the reduction of equations of the third and fourth degrees is attended with no little difficulty. So complicated is the solution of a complete equation of the fourth degree, that

algebraists try to avoid it when possible. Perhaps little is to be gained by the complete solution of complete equations of the fifth and other degrees, since methods of solving equations having numerical coefficients are well known.

My object in this chapter is to help the learner find the real roots of numerical equations.

The typical equation is

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \ldots = 0.$$

In this equation A, B, C, D, n are integral and all the exponents positive.

Suppose that an equation needs to be changed to the typical form, some or all of the following conditions may arise:

1. To make the exponents positive.

2. To make the exponents integral.

3. To make the coefficient of x^n unity.

4. To make the other coefficients integral.

Coefficients and Roots.

Let a, b, c, d be the roots of an equation of the fourth degree, then we have $(x-a)(x-b)(x-c)(x-d) = 0$; that is, $x - a = 0$, $x - b = 0$, $x - c = 0$, $x - d = 0$. It becomes

$$\left.\begin{array}{r|r|r|l|}
x^4 - a & x^3 + ab & x^2 - abc & x + abcd \\
-b & +ac & -abd & \\
-c & +ad & -acd & \\
-d & +bc & -bcd & \\
 & +bd & & \\
 & +cd & &
\end{array}\right. = 0.$$

The law is:

Coefficient of x^n equals 1.

Coefficient of x^{n-1} equals the sum of the roots taken with their signs changed.

Coefficient of x^{n-2} equals the sum of the products of the roots taken two at a time.

Coefficient of x^{n-3} equals the sum of the products of the roots taken three at a time with their signs changed.

The Absolute term is the product of all the roots. If the degree of the equation is odd, the sign of the absolute term is changed.

It will be seen that by an examination of the coefficients of an equation, frequently roots may be found, and then the degree of the equation depressed.

An excellent exercise is to give the roots of an equation to a class to form the equation.

First, let $a = 3$, $b = 4$, $c = 5$, $d = 6$. Secondly, $a = -3$, $b = 4$, $c = -5$, $d = 6$. Thirdly, take three of the roots negatively. Fourthly, all the roots negatively.

In each case, let the pupil or class note the differences in the signs of the coefficients.

Give exercises till these relations are well known and can be readily and quickly applied.

1. If any term is wanting, its coefficient is 0.

2. If the second term is wanting, the sum of the roots is 0.

3. If the third term is wanting, the product of the roots. taken two at a time is 0.

4. If the last term is wanting, one of the roots is 0.

5. The last term is divisible by each of the roots.

Exercises.

Given $x^3 - 3x^2 - 10x + 24 = 0$. Find the three roots by factoring the absolute term, 24. How many roots are minus? Why?

In the equation $x^4 - 12x^3 + 48x^2 - 68x + 15 = 0$, two of the roots are 3 and 5; how may the other roots be found? How is the depression of this equation effected?

Other Significant Properties.

1. Surd roots of the form $a \pm b$, or imaginary roots of the form $\pm b\sqrt{-1}$, enter equations by pairs; hence every equation of an odd degree must have at least one real root. Why?

2. In the equation $x^4 - 5x^3 - 7x^2 + 29x + 30 = 0$, how many positive roots are there? Why? How many negative roots? Why? What is the law for the number of positive roots in an equation? How can it be told by looking at an equation? If any term or terms be wanting, how tell them? Can the number of positive roots exceed the number of variations of signs? What must be the sum of variations and permanences in any equation?

3. If all the terms of an equation be positive, there is no positive root. Why?

Has the equation $x^4 - 11x^3 + 44x^2 - 76x + 48 = 0$ any negative roots? Why?

If all the terms of an equation be negative, how many positive roots are there? Why?

How many positive roots has the equation $x^7 - 1 = 0$? What is the test? What signs will be given to the wanting terms in the equation? If two or more successive terms of an equation be wanting, what inference in regard to imaginary roots?

4. **To determine whether an equation has equal roots.**

This is ascertained by finding the greatest common divisor of the equation and its first derived polynomial, or differential equation. If there is no common divisor, the equation has no equal roots. Suppose that in testing for equal roots a divisor of the form $(x - 1)^3(x + 3)^5$ is found; then $(x - 1)^3$ indicates three roots equal to 1, and five roots equal to -3.

5. **Superior and inferior limits of the roots of an equation.**

Determining these limits may not assist a great deal in

finding the roots of an equation, yet they are guides beyond which the operator may not look. To know that the greatest negative coefficient increased by unity is greater than the greatest root of the equation, is some knowledge; if not very definite, yet it is worth knowing.

Let us take the equation

$$x^5 + 5x^4 + 2x^3 - 14x^2 - 26x + 10 = 0$$

to find the superior limit of the superior roots. By formula

$$L = 1 + \sqrt[n-m]{P}.$$

n equals degree of the equation, m equals the degree of x in the highest negative term, and P the greatest negative coefficient. In this problem

$$n = 5, \quad m = 2, \quad P = 26;$$

$$\therefore\ L = 1 + \sqrt[5-2]{26} = 1 + \sqrt[3]{26}.$$

To find the limits of the negative roots, substitute $-x$ for x, and proceed as in the case of finding the positive roots.

Let the teacher select a group of equations such as the following:

1. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$,
2. $x^3 - 13x^2 + 56x - 80 = 0$,
3. $x^4 - 6x^3 + 5x^2 + 12x = 0$,
4. $12x^4 + 55x^3 - 68x^2 - 185x + 150 = 0$,

and question the class on each equation, thus:

How many roots has equation (1)? What is their product? What is their sum? What numbers may be its roots? Why? What are the limits of its roots? Has it an even or odd number of real roots? How many of its roots are positive? How many are negative? Has it equal roots? Why? Has it imaginary roots? Changing the

sign of every other term in the equation, what is the effect on the roots? If the signs of all the roots of an equation be changed, how is the equation affected?

Sturm's Theorem.

This theorem is employed when it is desirable to ascertain the *number* and *situation* of all the real roots of an equation. The equation should be freed from equal roots, but if it be not freed from the equal roots, it will still give the number of distinct roots without repetitions between the assigned limits. Sturm's Theorem is used chiefly in finding the situation of the incommensurable roots of numerical equations.

Some Properties of this Theorem.

1. No two consecutive functions can become zero for the same value of x.

2. When any function after the first vanishes, the two adjacent ones have opposite signs.

3. If, as x increases, $f(x)$ passes through zero, Sturm's functions lose one change of sign.

4. If any of the other functions vanish, there is neither loss nor gain in the number of changes of signs.

5. The total number of roots of $f(x)$ will be found by substituting $+\infty$ and $-\infty$ in the first term of each of the functions.

6. If the first terms in all the functions after the $f(x)$ are positive, then all the roots are real.

7. If the first terms are not positive, then for every change of sign there are two imaginary roots.

Horner's Method.

I will introduce this method by quoting an extract from Prof. Augustus De Morgan, who did so much to popularize Horner's discovery in England:

"Another instance of computation carried paradoxical

length, in order to illustrate a method, is the solution of $x^3 - 2x = 5$, the example given of Newton's method, on wihch all improvements have been tested.

"In 1831, Fourier's posthumous work on equations showed 33 figures of solution, got with enormous labor. Thinking this a good opportunity to illustrate the superior method of W. G. Horner, not then known in France, and not much known in England, I proposed to one of my classes, in 1841, to beat Fourier on this point, as a Christmas exercise. I received several answers agreeing with each other, to 50 places of decimals. In 1848 I repeated the proposal, requesting that 50 places might be exceeded: I obtained answers of 75, 65, 63, 58, 57, and 52 places. But one answer, by Mr. W. Harris Johnston, of Dundalk, and of the Excise Office, went to 101 decimal places."

Again I quote from De Morgan:

"It was somewhat more than twenty years after I had heard a Cambridge tutor show some sense of the true place of Horner's Method, that a pupil of mine who had passed on to Cambridge was desired by his college tutor to solve a certain cubic equation—one of an integer root of two figures. In a minute the work and answer were presented, by Horner's Method. 'How!' said the tutor; 'this can't be, you know.' 'There is the answer, sir,' said my pupil, greatly amused, for my pupils learnt not only Horner's Method, but the estimation in which it was held at Cambridge. 'Yes,' said the tutor, 'there is the answer, certainly; but it stands to reason that a cubic cannot be solved in this space.' He then sat down, went through a process about ten times as long, and then said with triumph: 'There! that is the way to solve a cubic equation.'"

The discovery of this method and its application to the solution of numerical equations of the higher degrees, as a labor-saving device, is excelled only by Logarithms. In the solution of numerical equations, unless I can reduce them

by some tentative process or by factoring, I use Horner's Method invariably. It can be taught to a class in Arithmetic just after 'Cube Root' is learned."

To teach it in Algebra, give a cubic wanting the second and the first power of the unknown quantity. Then when this class of problems can be easily solved, introduce a complete cubic of this form, $x^3 + x^2 + x = 6{,}000$; to be followed by cubics of still greater difficulty.

The same plan should be pursued in the solution of equations of the fourth degree. About the only difficulty the learner experiences is in placing the numbers in their proper places in the columns each time.

Both in cubics and biquadratics, the learner should find the negative roots as well as the positive ones.

Proceeding in a precisely similar manner, take up higher equations. It is not till the pupil understands how to find the number and the situation of the roots of any numerical equation, and then is able to solve it, that he is prepared to handle algebraic problems. It is a fitting climax to elementary Algebra for the learner or class to be thoroughly grounded in the most important principles involved in the philosophy of equations.

Loci of Equations.

An interesting phase of Algebra is to represent equations by diagrams or figures. While the subject belongs properly to another branch of mathematics, its introduction has a tendency to lead the learners onward to a higher conception of the equation. As soon as they discover that an equation of the first degree is the equation of a straight line, and that an equation of the second degree represents a curve, they naturally look upon an algebraic equation as an expression whose properties can be revealed, constructed, and interpreted.

If the teacher desires his class to construct the *loci* of some equations, the following terms need to be learned :

1. The Axes of Reference.
2. Their Origin.
3. The Axis of Abscissas.
4. The Axis of Ordinates.
5. The Abscissa of a Point.
6. The Ordinate of a Point.
7. The Fort of an Ordinate.
8. The Coördinates of a Point.
9. The Locus of an Equation.
10. Constructing its Locus.
11. Abscissas—positive and negative.
12. Ordinates—positive and negative.
13. The names of the parts into which the plane is divided by the axes.
14. Counting Directions.

The first step is to teach the learner to draw the axes, and then how to locate points in the four quadrants.

2d. To construct such equations as

$$y = 2x + 2, \qquad x = 2y - 3, \qquad y = mx + b.$$

3d. $4x - 5y = 5, \quad 3x + 2y = 21,$
$6x + 12y = 78. \quad 2x + 12y = 40.$

4th. $x + y = 9$ and $x^2 + y^2 = 53,$
$x + y = 7$ and $x^2 + 2y^2 = 34,$
$x - y = 10$ and $x^2 + y^2 = 178.$

5th. $x^2 + 3x - 10 = 0, \quad x^3 - 2x^2 + 1 = 0,$
$x^4 - 20x^2 + 64 = 0.$

Geometry.

Historical Sketch.

The origin of geometry is hidden in the past. The derivation of the word, from the two Greek words, "ge" and "metron," signifies "earth-measuring," or, more popularly, "land-measuring." Herodotus says, in speaking of Sesostris, King of Egypt: "They said also that this king divided the country amongst all the Egyptians, giving an equal square allotment to each, and from this he drew his revenues, having required them to pay a fixed tax every year; but if the river happened to take away a part of any one's allotment, he was to come to him and make known what had happened; whereupon the king sent persons to inspect and measure how much the land had diminished, that in future he might pay a proportionate part of the appointed tax. Hence land-measuring appears to me to have had its beginning, and to have passed over into Greece; for the pole and the sun-dial, and the division of the day into twelve parts, the Greeks learnt from the Babylonians."

Diodorus re-enforces this statement as follows: "The river, changing the appearance of the country very materially every year, causes various and many discussions among neighboring proprietors about the extent of their property; and it would be difficult for any person to decide upon their claims without geometrical proof."

Rouché and De Comberousse, in the preface to "Traité de Géométrie Élémentaire," maintain that the ideas of extent, position, and form are as ancient as the race, and

they attribute to the Egyptians and the Chaldeans the first attempt to co-ordinate these ideas.

Thales of Miletus, in Asia Minor, born about 640 B.C., introduced geometry into Greece from Egypt, where he had been instructed by the priests. Upon his return to Greece he founded the Ionian School of Philosophy. He predicted the eclipse of the sun which occurred during a battle between the Medes and Lydians about the year 609 B.C. He discovered that all angles in a semicircle are right angles, and he demonstrated some propositions relating to the similarity of triangles. The height of the pyramids he measured from their shadows, and by an application of the principles of geometry he could tell the distance of vessels remote from the shore. Many of his propositions were afterward collected in Euclid's Elements.

Pythagoras of Samos, a disciple of Thales (580 B.C.), founded a celebrated school in Italy that bore his name. He had studied in Egypt, spent some time in Babylon, and perhaps visited India prior to his residence in Italy. To him are attributed the discovery of the incommensurability of the diagonal and the side of a square, that the square on the hypothenuse of a right-angled triangle is equivalent to the sum of the squares on the other two sides, that the circle has the maximum area of any plane figure having the same perimeter, and the sphere the maximum volume bounded by a given surface. He was the first to investigate the properties of regular polyhedrons, and one of his pupils solved the problem of finding two mean proportionals between two given straight lines.

Hippocrates, of the island of Chios, who lived about 400 B.C., was one of the most noted Greek geometers of antiquity. He was the first to effect the quadrature of a curvilinear space by finding a rectilinear space equivalent to it; and he demonstrated that the crescent bounded by half the circumference of one circle and one fourth the

circumference of another is equal to an isosceles right-angled triangle whose hypothenuse is the common chord of the two areas; also, that the duplication depends upon finding two mean proportionals between two given lines.

Plato, the philosopher, was one of the most distinguished promoters of the science. He introduced the analytical method of investigation, and discussed some properties of conic sections and of geometrical loci. In the school which he established, the *duplication of the cube* and the *trisection of an angle* were first investigated. The first of these two noted problems Plato solved; but the second has thus far baffled all attempts at a solution by elementary geometry. Eudoxus, who lived at the same time with Plato, found the volume of the pyramid and cone, and made considerable proficiency in conic sections.

Plato's disciples gave a great impulse to the science of geometry. Euclid, who belonged to the famous school of Alexandria, had studied at Athens under the followers of Plato. This remarkable school was established about 300 B.C. Perhaps fifteen years later, Euclid collected and systematized all the truths, propositions, and theorems then known as elementary geometry, and to which he added many new ones. This treatise is known as "Euclid's Elements." In reply to King Ptolemy, who had asked if there was no easier way to learn geometry than in his "Elements," Euclid promptly said: "*No, sir; there is no royal road to geometry.*" The method of proof known as the *reductio ad absurdum* first appeared in "Euclid's Elements." Many of Euclid's writings were lost, the most important one of which was his treatise on prisms. The "Elements" is the work that is best known. It is composed of thirteen books, which treat of geometry and arithmetic. This work has been translated into the languages of all nations that have made much progress in the sciences and arts, and has been more generally used in

teaching than any other text-book ever written. Of the thirteen books composing "Euclid's Elements," the first four and the sixth treat of figures in a plane, while the fifth treats of proportion; the next four belong to arithmetic, and incommensurable quantities. The eleventh and twelfth treat of solid geometry, and the thirteenth book relates to the five regular solids. Two additional books, once credited to Euclid, were doubtless added about two hundred years later. School editions usually embrace the first six and the eleventh and twelfth books.

Archimedes, born at Syracuse 287 B.C., one of the most distinguished geometers of antiquity, wrote two books on the sphere and cylinder. He demonstrated that the sphere is two thirds of the circumscribing cylinder, whether their surfaces or volumes be compared. He determined the circumference of a circle to be between $3\frac{10}{70}$ and $3\frac{10}{71}$ times greater than its diameter; he compared the area of the ellipse with that of the circle, and proved that the area of any segment of a parabola cut off by a chord is equivalent to two thirds of the circumscribing parallelogram. His line of work was in the direction of metrical geometry.

Apollonius was a profound and original geometer, born at Perga in Pamphylia about 250 B.C. His writings relate to the geometry of form, the conic sections in which he developed the properties of asymptotes, foci, conjugate diameters, normals, the theory of polars, and the primary notions of maxima and minima; and the celebrated theory of cycles and epicycles, so long employed in explaining the apparent movements of the heavenly bodies, is attributed to him. His principal work, the treatise on conic sections, contained eight books, seven of which are still in existence. This work was written in Greek during his residence at Alexandria. An Arabic version of one of his treatises is preserved. The title of "Great Geometer" was given him.

The successors of Archimedes and Apollonius directed their studies to those sciences which had a particular bearing on the science of astronomy. Also, about 150 B.C. flourished Hipparchus, the great astronomer of antiquity, who is regarded as having discovered the method of projecting the sphere stereographically, and of having invesigated the properties of transversals in both plane and spherical triangles.

From this date till the discoveries of Pappus, who lived at Alexandria about 380 or 400 A.D., geometrical investigation had virtually ceased. Pappus announced the principle of the famous rule now known as "Guldin's Theorem;" he gave the first example of the quadrature of a curved surface; the fundamental principles of the anharmonic relation; the germ of the theory of involution; and the property of the hexagon inscribed in a conic section.

Hypatia, the celebrated daughter of Theon, displayed even greater talents than her father, whom she succeeded as teacher of mathematics at Alexandria near the close of the fourth century. She wrote commentaries on Apollonius and Diophantus. Her works were destroyed when the Mohammedans burned the library of Alexandria. The Alexandrine school ceased when the city was conquered by the Arabs. With the fall of Alexandria in 638 A.D., another school sprang up at Bagdad. A few able commentators had access to some writings of the Greeks which had escaped the disastrous conflagration at Alexandria; but in Europe a profound stagnation prevailed for a thousand years, which clearly divides the ancient geometry from the modern. After the revival of learning "Euclid's Elements" were first made known in Europe through the medium of an Arabic translation.

Vieta, the veritable creator of algebra, applied this science to the solution of problems in geometry. He constructed graphically the roots of equations of the second and third degrees, and was the first to solve the problem

of drawing a circle tangent to three given circles; but the modern methods of solution are more elegant and simple. We are indebted to Vieta for the new and fruitful idea of the transformation of spherical triangles; and his reciprocal triangle, without doubt, conducted Snellius to the discovery of the supplementary triangle. The writings of Kepler (1571–1631) and of Fermat (1570–1633) contain the germs of the method of infinitesimals. We owe to Kepler, the founder of modern astronomy, the treatment of the circle as composed of an infinite number of triangles, having their vertices at the center, and the cone as composed of an infinite number of pyramids, all having the same vertex as the cone; and to Fermat, the restriction of the plane surfaces of Apollonius, and the first complete solution of the problem relating to the contact of spheres.

Pascal, so well known on account of his works on the cycloid, indivisibilities, and the calculus of probabilities, discovered, at the age of sixteen, the beautiful properties of the "mystic hexagramme," or Pascal's Theorem, which he took for the basis of his complete treatise on conics. A skeleton or outline of his works, as an essay on the conics, was published in 1640. From the writings of Pascal is recognized the influence exerted upon his contemporary Lyonnais Desargues (1593–1662), who was one of the most skillful geometers of that age, and whom M. Poncelet called the *Monge of the century.* The ancient geometricians studied the conic sections from the cone itself, and employed tedious solutions for each of the curves. Desargues referred directly all properties of the conics to the circle at the base of the cone, regardless of what the forms might be. Among the discoveries he made were the inscribed quadrilateral in a conic, the fundamental properties of homologous triangles, and an extension of the properties of the circle so as to include all classes. His demonstrations were broad generalizations.

Descartes was born in 1586 and died in 1650; he produced a complete revolution in geometrical methods by bringing geometry under the domain of algebraic treatment, thus founding that branch of mathematical science called *Analytical Geometry*. Owing to the universality of his solutions and the comprehensiveness of their scope, the ancient method fell into comparative neglect among the mathematicians of the Continent with few exceptions, but as a method of discipline it was encouraged somewhat extensively in the English schools. Among the more illustrious names who maintained the excellence of the ancient Greek geometry may be mentioned Huygens (1629–1695) and La Hire (1640–1718) on the Continent, and Newton in England. To La Hire we owe the discovery of the theory of poles and polars. The discoveries of Leibnitz and Newton in the infinitesimal calculus diverted attention for a time from pure geometry. Newton showed, however, that it could be employed in the higher branches of investigation. Two English mathematicians, Cotes and Maclaurin, applied their methods to the investigation of geometrical curves. The astronomer Halley (1656–1742) by his beautiful translations of Apollonius, and Robert Simson in his writings on the conic sections and prisms, endeavored to revive a taste for the ancient geometry; but their efforts were only instrumental in keeping alive the interest among a limited number of analysts in Great Britain and on the Continent. Little progress had been made till the brilliant discoveries of Monge and Carnot at the beginning of the present century. Gaspard Monge, the creator of descriptive geometry, was born at Beaune, France, in 1746, and died in 1818. His first edition of descriptive geometry was published in 1795; he also left another important work—"Application of Analysis to Geometry." His discoveries mark an epoch in the science of geometry, for which he did more than any other writer since Archimedes.

Carnot, seven years younger than Monge, and a pupil of his, enriched the science by his "Geometry of Position" and his "Essay on Transversals." Monge's deductions showed the intimate relations between plane figures and figures in space; or that from the properties of bodies of two dimensions corresponding properties of bodies of three dimensions could be deduced. These relations gave rise to many new and elegant theorems. Carnot's discoveries reached similar conclusions by pure geometry that Descartes had obtained by the analytical method. Arriving at the same results by different processes of investigation still further illustrated the vast possibilities of mathematical research. To the splendid discoveries of Monge and Carnot should be added those of Poncelet (1788–1867) in his remarkable treatise on "The Properties of the Projection of Figures," in which he employed the principle of continuity and the beautiful theorem of reciprocal polars and of homologous figures to demonstrate all known properties of lines and surfaces of the second order.

This brief sketch will close with a reference to the great works of Chasles (1793–1880) on higher geometry, his treatise on prisms, researches on the attraction of ellipsoids, cones of the second order, ruled surfaces, and a memoir on duality and homography; a new method of determining the characteristics of systems of conics, and other productions of this eminent master. The investigations in geometry are still progressing under the keenest analytical skill of two continents, and no one yet has fixed a limit—*thus far and no farther.*

Elementary geometry in this country is based upon "Euclid's Elements" as translated and modified by Dr. Robert Simson and improved by subsequent writers, or upon the treatise by Legendre, published first in 1794; but Legendre's text has been greatly improved by different editors. However, the most complete work that I have

ever examined is "Traité de Géométrie Élémentaire, par Eugène Rouché et Ch. De Comberousse," Paris, 1868. This great work is divided into two parts,—*Plane Geometry* and *Geometry in Space.* Plane Geometry fills 328 pages, and Geometry in Space 472 pages.

For a full and complete presentation of the life and writings of Euclid, and a history of the translations of his "Elements," the reader is referred to the article "Eucleides of Alexandria," by Professor Augustus De Morgan, Vol. II., Smith's Dictionary of Greek and Roman Biography and Mythology.

TEACHING GEOMETRY.

Primary Conceptions.

The final object in teaching geometry is to make good geometricians. Clear notions, sharply cut and accurately defined, must be fairly planted in the learner's mind. From the beginning he must know what he knows, and he must know what he does not know. It is assumed that the learner has mastered arithmetic and enough of algebra to enable him to pursue the subject of geometry intelligently. For those teachers who are desirous of starting younger pupils in geometry, the Primary Treatises by George Spencer and Dr. Thomas Hill are the best, perhaps, published. However, the child of average ability has picked up, in one way or another, considerable geometrical knowledge before he begins the subject in earnest. His knowledge is crude and unorganized.

In teaching classes, or a single pupil, the first thing is to teach a clear conception of a limited portion of space. A crayon-box will answer the purpose well. It is held before the class. All see it, and can tell how many sides, ends,

corners, and edges it has. They see, too, that it occupies a definite portion of space. Let the teacher remove it, and see if the members of the class can think of the definite portion of space that the box formerly occupied. This brings the class to the conception of a portion of pure space, and from which the box has been removed. They need not follow the track of the box through space yet from one place to another; but their minds should be held to the contemplation of the "box-space" until it becomes a permanent notion. Now, let the box be brought before the class again. Tell the class that the top and bottom will remain the same size; but that the two side-pieces and end-pieces will gradually and evenly shrink away until the top and bottom approach each other and become one thin piece perfectly smooth and level. Question the class on this form. Next, assume that the ends begin to approach each other until they coincide, and question as before. If these changes in space can be readily followed by the class, assume the regular contraction of the sides and ends of the box at the same instant till they vanish. Upon the other hand, let the box begin to expand regularly till its length is ten feet, and the ends are enlarged proportionately. This last space may now be discussed by the members of the class. If necessary, the sides and ends may be assumed as having almost no thickness, and then contracted or expanded at pleasure, and the class questioned till the conception of a "hedged-in portion of space" within space is a reality in the mind. Instead of an oblong box, a cubical block may be taken at first, or any other convenient object. The object to be accomplished through illustrations is that of pure form; or in other words, the notion of the space a body occupies without respect to the material composing the body itself. This conception is the true starting-point in geometry, and

it should be gained first through sensuous forms. The time required varies for different learners.

The next step in the process is to take a definite surface, say, the top of the crayon-box, and let the class conceive the ends to contract regularly, the length of the sides remaining unchanged, until the two sides become one straight line; that is, the surface changes by contraction into a straight line. Use other illustrations, if necessary, to fix this notion. From the material line, however made, the process is to be continued until a conception of a pure geometrical line, as one of the boundaries of a pure solid, is obtained.

Appropriate questions will enable the teacher to determine this fact, and to correct any erroneous impressions the learner may have on the subject. A plane surface thus changes into a straight line, and by thinning the surface it is made to approach a pure geometrical line. By still further refining, the material line approaches, in thought, indefinitely near to the pure line in space with which geometry deals. Since the notion of a line can be thus derived from the surface or lid of the box, the line may be considered as shortening till it becomes a point. The successive steps are *solid, surface, line,* and *point,* as obtained through the material forms. Putting aside these crude notions, the pupil or the class must now pass to the space conceptions of the *solid, surface, line,* and *point.* The geometrical solid in pure space can now be considered without regard to the material form from which it was derived.

Two of these surfaces that intersect form a *line,* and when three of them form a corner, it is a *point.*

It is necessary for the pupil to obtain clear conceptions of a *geometrical solid, surface, line,* and *point* at the outset.

This process, as thus indicated, is essentially analytic; it

needs to be reversed before the pupil apprehends it in all its force and beauty. Suppose he now assumes a *geometrical point.* This point is supposed to move in space. *Its path is a line.* If it move in one direction, its path is a straight line; hence from a *point* a line is generated. Suppose this *line straight* and ten inches in length, and it moves ten inches parallel to itself, thus describing a surface whose area is a hundred square inches. It is readily seen that a line can be moved in space so as to describe a surface. Next move the surface, or plane, ten inches parallel to itself, and form a cube; hence a surface may be made to describe a solid.

Geometry proper is the science of *position, extent,* and *form;* or more concisely, it deals with *form abstractly.*

All material bodies occupy a limited portion of space; hence they have position, extent, and form; i.e., each body is *somewhere, has some size,* and is of *some shape.*

To be Illustrated by the Pupil.

Show that a solid has length, breadth, and thickness. Show that a plane surface has length and breadth. What are the edges of a page of writing-paper? If two pieces of pasteboard cut each other, what does their intersection represent? If two lines intersect, what do they form? In a cube, how many faces meet in an edge? In a corner? Show that the path of a moving point is a line. Show that the path of a moving line is a surface. When is it not a surface? Why? When is the path of a moving surface a solid? What exception is there? Why?

Given a point; derive from it a line, a surface, a solid. Given a solid; reduce it to a point. What are the limits of a solid? The limits of a surface? The limits of a line?

These questions are merely suggestive; but they indicate the plan to be pursued.

Definitions.

Mathematics, as an abstract science, rests substantially upon a few well-digested definitions. The key to geometry lies in its definitions. These must be mastered at the beginning; otherwise successful progress is impossible.

1. A solid is extension having length, breadth, and thickness.

2. A surface is extension having length and breadth.

3. A line is extension having length.

4. A point is position only.

Explanations.

5. A Solid may be regarded under two conditions: 1. As a path formed by a moving plane; 2. As extension, having length, breadth, and thickness.

6. A Surface may be regarded under three conditions: 1. As the limit of a solid; 2. As the path formed by a moving line; 3. As extension, having length and breadth.

7. A Line may be regarded under five conditions: 1. As a limit of a surface; 2. The intersection of two surfaces; 3. The path of a moving point; 4. As the assemblage of all the positions of a generating point; 5. As extension in the direction of length.

8. A point may be regarded under three conditions: 1. As one of the limits of a line; 2. The intersection of two lines; 3. As position only.

Postulates.

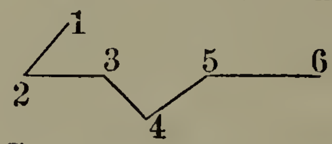
1. A magnitude can have any position.

2. A magnitude can have any form.

3. A magnitude can have any extent.

Classification of Lines.

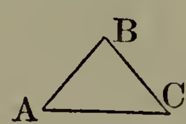
1. Straight Lines, as : 

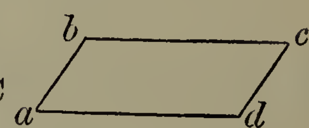
2. Broken Lines, as : 

3. Curved Lines, as : 

4. Mixed Lines, as : 

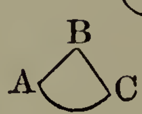
Classification of Plane Surfaces.

1. Rectilinear Surfaces, as: 



Etc.

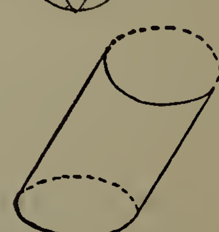
2. Curvilinear Surfaces, as: 

3. A Mixed Surface, as: 

Classification of Solids.

1. Solids bounded by plane surfaces, as : 

2. Solids bounded by curved surfaces, as : 

3. Solids bounded by mixed surfaces, as : 

General Definitions.

1. **An Axiom** is a self-evident truth.

2. **An Absurdity** is a self-evident falsity.

3. **A Postulate** is a self-evident possibility. It states that something can be done, but does not tell how.

4. **A Theorem** is a truth to be proved.

5. **A Problem** is a question prepared for solution.

6. **A Proposition** is the expression of a judgment.

7. **Theorems,** problems, axioms, and postulates are called propositions.

8. **A Formula** is a theorem expressed in algebraic language.

9. **A Corollary** is an obvious truth deduced from the proposition to which it is attached.

10. **A Scholium** is a remark upon some part of a proposition.

11. **A Lemma** is an auxiliary theorem to be used in the demonstration of another proposition.

12. **A Demonstration** is proof of a proposition.

13. **Demonstrations** are of two kinds: 1. *Direct;* 2. *Indirect.*

14. **A Direct Demonstration** proves a proposition in either of two ways: 1. *By Superposition;* 2. *By logical combination* of definitions, axioms, and previously demonstrated propositions.

15. **An Indirect Demonstration,** called also *Reductio ad absurdum,* proves a proposition true by showing that the supposition that it is false involves *an absurdity.*

16. **The Converse** of a *categorical or disjunctive* proposition is obtained by interchanging the subject and predicate. Thus, No A is B; conversely, No B is A. Also, A is B or C; *conversely,* B or C is A.

17. The Converse of a hypothetical proposition is ob-

tained by making the hypothesis of the original proposition the conclusion, and the conclusion the hypothesis. Thus, if A is B, it is not C; *conversely,* if A is C, it is not B.

18. **The Solution** of problems or exercises in Geometry consists of four parts: 1. The Analysis or course of thought by which the construction is found out. 2. The Construction of the figure with the aid of compass and ruler. 3. The proof. 4. The discussion or limitations of the problem.

19. **Similar Magnitudes** are those that have the same form.

20. **Homologous points,** lines, or surfaces, are similarly situated points, lines, or surfaces in similar magnitudes.

21. **Equivalent Magnitudes** are those that have the same extent.

22. Equal magnitudes are such that one can be applied to the other, and they coincide.

23. **Superposition** is to apply mentally one magnitude to another.

Questions for Review.

1. What is meant by a *Definition?*
2. When is a definition *redundant?* When *deficient?*
3. What is the literal meaning of the word *definition?* From what word is it derived?
4. From what language is the word *Axiom* derived? *Absurdity? Postulate? Theorem? Problem? Proposition? Judgment? Expression? Formula? Corollary? Scholium? Lemma? Demonstration? Superposition? Hypothesis? Categorical? Hypothetical? Solution? Magnitude?*

Show in what respects each of the foregoing words varies from its original meaning.

Remark. Pupils should be taught how to find these

words, and all similar terms, in the Unabridged Dictionary, and to trace out their meaning either from the Latin or the Greek. If the pupils have not learned the Greek Alphabet, they can do so in an hour or two. A severe drill in "Etymology," when Latin and Greek have been omitted, will enable pupils to resolve words with ease and pleasure. A word should be resolved into its etymological elements. For instance, the word "surface," derived from the Latin *super*, above, and *facies*, a face, has a different meaning in Geometry from the same word applied to a region of country.

Lines and Angles.

1. **Parallel lines** are straight lines everywhere equally distant. They lie in the same plane.

2. **An angle** is the difference in direction of two intersecting lines or of two intersecting planes.

BAC is a plane angle; *AB* and *AC* are the sides, and the point *A* is the vertex. The difference in direction of the two sides is the magnitude of the angle *A*.

3. Angles are divided into two classes: 1. *Right;* 2. *Oblique,* and the oblique into the *Acute* and *Obtuse.*

4. **An angle is measured by the space through which one of the sides must turn in order to concide with the other side.**

While one side remains fixed in the plane, the other may revolve once around the vertex in the plane, thus forming an angle of 360 degrees; or twice, forming an angle of 720 degrees; and so on: but if in one direction the angle is regarded as positive, in the opposite direction it will be negative.

Euclid permits the student of Geometry to use a ruler and a pair of compasses to construct the propositions to be demonstrated. Instead, the pupil can use a straight-edged

ruler, a string, and a piece of crayon for blackboard work, and a ruler and a circle-pen for paper diagrams.

Postulates.

1. A straight line can be drawn from any point, in any direction, to any extent.
2. A straight line can be drawn from any point to any other point.
3. A straight line passing through two points is fixed in position.
4. A straight line can be produced indefinitely in either direction and to any extent.
5. A circumference can be described from any center and with any radius.

Suggestions.

The definitions and explanations have been stated in full, and for the reason that the learner must lay a good foundation to succeed in the prosecution of the subject, Each technical term conveys a distinct idea, and these are the tools that the learner employs in his work; hence the importance of critically examining each, and ascertaining that it is in proper condition for immediate and successful use. To learn Geometry well means persistent, intelligent, and severe mental application. The subject may be taught in such a manner as to blunt the intellects of the pupils; or, on the other hand, to stimulate them to the very highest degree of intellectual enthusiasm.

The conditions for successful work are (1) active, energetic, and wide-awake pupils; (2) a good text-book with plenty of exercises for pupils; (3) ruler, string and crayon, and blackboard surface; or, pencil, ruler, circle-pen, and paper; (4) a first-class teacher, who understands the subject and can put spirit into the work.

Rectilinear Figures.

Under Book I. the following classification is adopted:

1. Definitions and general principles.
2. Perpendiculars and oblique lines.
3. Parallel lines.
4. Triangles.
5. Quadrilaterals.
6. Polygons in general.
7. Exercises.

Definitions and General Principles.

Whatever definitions are placed at the beginning of a book in Geometry are to be learned, illustrated, and assimilated. Committing definitions without understanding their meaning is of no educational or mathematical value whatever, except in those special cases in which the learner is blessed with a very retentive memory, which will hold them long enough for actual knowledge to be acquired and classified under the definitions. Under this head it is important that the teacher should train his pupils to give the three forms of definitions:—1. *Nominal* or root definition; 2. *Real definition*; 3. *Genetic definition.*

When a term is used in a technical sense, this distinction should also be noted, and the two meanings compared.

Perpendicular and Oblique Lines.

When the pupil is called upon to demonstrate a theorem, (1) he must state exactly what he proposes to do; in other words, *he states the question;* (2) he proceeds to prove what he has said; (3) he must confine himself strictly to the subject; (4) he must decide whether he has proved what he undertook; (5) he must resolve the conclusion back to primary conceptions. The pupil should have within himself the

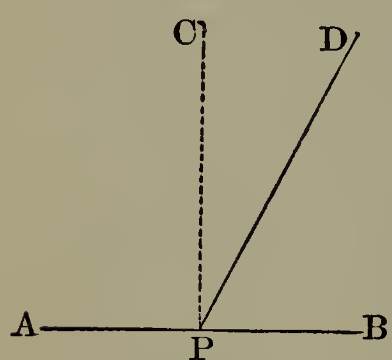
means of testing his own work. Geometry as usually taught is barren of results, because the learner does not apply what he learns, and consequently it is soon forgotten. As an illustration, suppose the pupil is required to demonstrate that—

"At a given point in a straight line one perpendicular to the line can be drawn, and but one."

After the proposition is assigned, the pupil will construct the figure. Next, he formally states what is to be proved. Then he proceeds with the proof. This concluded, he discusses all possible limitations that can be imposed upon the question. Original demonstrations, corollaries, and remarks add sprightliness to all exercises.

Steps.

1. *Construction.* In the annexed diagram let AB be the straight line, P the given point, CP the perpendicular, and PD any other oblique line to AB at the point O.



2. *Demonstration.* Suppose AP to remain immovable while the line BP begins to revolve in the plane of the paper about the point P as an axis. When the point B has reached D, the angle BPD is increasing and the angle APD is decreasing; consequently there is one point at which the angle BPD is equal to the angle APD. For the angle BPD was zero when PD coincided with PB, and when PD passes over to AP, then the angle APD is zero; hence there is one position, as CP, when the two angles are equal.

3. *Conclusion.* "At a given point,' etc."

4. *Discussion.* The theorem is not properly *restricted.* An infinite number of perpendiculars can be drawn from a point in a straight line. Suppose the perpendicular CP

to revolve about the point P as an axis, and in every position being perpendicular to AB, it is evident that an indefinite number of perpendiculars can be drawn from a point in a straight line thus revolving. To limit the theorem properly, should be added "in a plane embracing the line."

5. *Questions.* If the angle $BPD = 60°$, what is the angle DPC? What is the complement of the angle BPD? Of the angle CPD? What is the supplement of the angle BDC? What is the angle that the line AP makes with the line BP at the point P? Prove that the right angle APC is equal to the right angle BPC. Prove that the complements of equal angles are equal. That the supplements of equal angles are equal. If the angle APD is equal to $115° \, 30'$, what are the values of the angles BPD and DPC respectively? How many degrees are there in the complement, and in the supplement, of $\frac{3}{5}$ of a right angle? How many degrees are there in an angle whose complement is five times the angle? If the supplement of an angle is five times the angle, what is the complement of the angle? How many degrees are there in the angles $BPD + DPC + CPA$?

Another proposition will be given to indicate still further the method of teaching.

THEOREM. **The two adjacent angles which one straight line makes with another are together equal to two right angles.**

1. *Construction:* In the diagram AB and PC are the two straight lines which meet at the point P.

2. If the angle APC is equal to the angle BPC, then the two angles are right angles, and are equal by definition, and CP will coincide with DP.

If they are not equal, let DP be drawn perpendicular to AB, and then the angle APD is equal to the angle BPC plus the angle CPD.

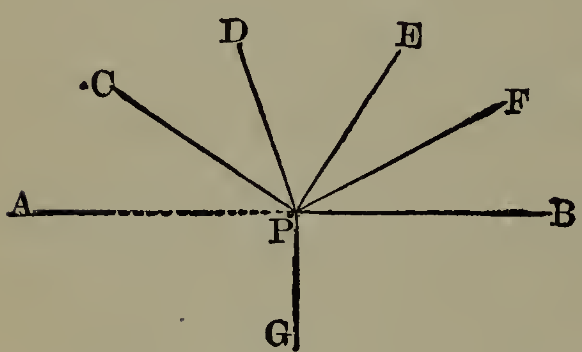
But the angle BPC plus the angle CPD are equal to a right angle; hence the angle APD plus the angle DPC

plus the angle CPD are equal to two right angles, because the angle DPC plus the angle CPB are equal to the angle DPB.

1. Corollary. **The sum of all the consecutive angles formed by any number of straight lines in the same plane, drawn from the same point of a straight line on the same side, is equal to two right angles. Prove.**

2. **When the sum of the successive angles at a point in a line is equal to two right angles, the two extreme arms form a straight line.**

3. **The sum of all the angles formed by straight lines in a plane, meeting at a common point, is equal to four right angles. Prove from the annexed figure.**



If $\angle APC = 27° \ 30'$, $\angle CPD = 36°$, $\angle DPE = 38° \ 30'$, $\angle FPB = 41° \ 45'$; how many degrees in the $\angle EPF$?

If $\angle APG = 82° \ 29' \ 30''$, how many degrees in the $\angle GPB$?

4. **Angles may be added or subtracted. Illustrate.**

5. **The greater an angle the less its supplement.**

6. **Vertical or opposite angles are equal.**

Prove from the annexed figure that the angle APC is equal to the angle BPD; also, that the angle APD is the supplement of either the angle APC or the angle BPD.

Definitions that are Constantly Used.

The sooner the learner familiarizes himself with the following geometrical terms and their meaning, the more intelligent and satisfactory will be his progress. He must accustom himself to think in geometrical language.

1. Exterior and Interior Angles.
2. Exterior angles of a Triangle.

3. Alternate Angles.

4. Corresponding Angles.

5. A Transversal. Angles formed by a Transversal.

6. Bisector of an angle. Bisectors of vertical angles. Bisectors of a pair of vertical angles formed by two intersecting straight lines.

7. Bisector of an angle as a Locus.

Special importance should be attached to the *hypothesis, demonstration,* and *conclusion* of each proposition. Then if the proposition admits of a *converse* proposition, that also should be enunciated. By all means the learner must start right, must keep right, must learn thoroughly, and be able to give a valid reason for every step he takes.

As new terms are introduced, their etymological meaning should be critically investigated. Words are the instruments of thought.

Parallel Lines and Angles.

1. Two straight lines in one plane may be discussed under the following conditions:

(*a*) **If they are parallel, they never meet.**

(*b*) **If they can not meet, they are parallel.**

(*c*) **If they can meet, they are not parallel.**

(*d*) **If they are not parallel, they can meet.**

(*e*) **If they intersect, they form oblique or right angles.**

(*f*) **If one angle is oblique, all the other angles are oblique.** What exception?

(*g*) **If one angle is right, all the other angles are right.** What exception?

2. When two or more straight lines intersect in two or more points, but not forming an enclosed figure, the following cases deserve special notice:

(*a*) **When two or more straight lines are perpendicular to the same straight line, and conversely.**

(*b*) **When two parallel straight lines are cut by another straight line the alternate interior angles are equal; the alternate exterior angles are equal; the corresponding angles are equal; and the interior angles on the same side of the secant line is a constant quantity, equal to 180 degrees.**

The converse of these conditions in each case is true.

(*c*) **Two angles whose sides are parallel each to each, are either equal or supplementary.** Note the cases when the parallel sides extend in the same direction, or in opposite directions.

(*d*) **Two angles whose sides are perpendicular each to each are either equal or supplementary.** Note the cases if both angles are acute or both obtuse; if one is acute and the other obtuse.

The doctrine of parallel lines is a very important one. Parallel lines in Geometry may beregarded a s akind of *wheelbarrow,* upon which almost everything can be loaded and pushed along. Important propositions will be strongly accentuated.

Questions. Two straight lines intersect; one of the angles is 83° 44′ 33″: what are the other angles? Two parallel lines are cut by a third line; if one angle is 36°, what are the other seven angles?

Two straight lines meet in a point; they are cut by a third line; if one angle is known, formed by the third line and either one of the other lines, how many other angles can be found?

Illustrate each question by a diagram.

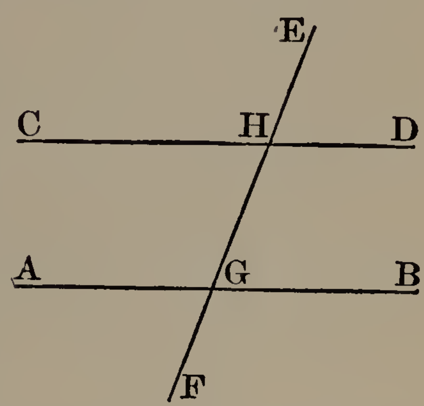
Demonstrations.

Demonstrations different from the ones employed in the text should be discovered by the pupils. Such demonstrations are more lasting, and make deeper impressions on the mind. If a demonstration is to be written out, it should be a model of neatness in arrangement and of logical ac-

curacy in reasoning. In no other species of composition is there so great necessity for precision in the use of language. As an illustration, suppose this familiar theorem: "If two parallel lines are cut by a third straight line, the corresponding angles are equal."

The construction is omitted.

1. *Demonstration.* Suppose the lines *CD* and *EF* to remain fixed, and let the line *AB* be moved parallel to itself toward *CD* till it coincides with the line *CD*, then the point A falls on *C*, *G* on *H*, and *B* on *D*; therefore the angle *HGB* coincides with the angle *EHD*, the angle *AGF* with the angle *CHG*, the angle *AGH* with the angle *CHE*, and the angle *FGB* with the angle *GHD*,—which prove that the corresponding angles are equal.



2. *Demonstration.* Conceive the angles at G transferred to H, the direction of the lines remaining unchanged; then each angle will coincide with its corresponding angle, and be equal to it.

Remark. Theorems may often be arranged in groups of four: 1. The original theorem; 2. Its opposite; 3. Its converse; 4. The converse of the opposite.

Example. "1. If two lines are parallel, the corresponding angles will be equal. 2. If two lines are not parallel, the corresponding angles will be unequal. 3. If the corresponding angles are equal, the lines will be parallel. 4. If the corresponding angles are unequal, the lines will not be parallel."

Exercise on Lines and Angles.

The test of the learner's knowledge is his ability to handle successfully original exercises. Under each theorem exercises should be given for practice.

1. Five straight lines in a plane meet at a point, making equal angles with one another around that point: how many degrees in each angle, and what part of a right angle is each angle?

2. Prove that the bisectors of adjacent supplementary angles are at right angles to each other.

3. Find the angle between the bisectors of adjacent complementary angles.

4. Ten lines meet at a point so as to form a regular ten-rayed star: what is the angle in degrees between two consecutive rays?

5. If A is the number of degrees in any angle, prove that $90°+A°$ is the supplement of $90°-A°$; and that $45°+A°$ is the complement of $45°-A°$.

Triangles.

The following terms are of frequent use, and if not already known, are now to be learned:

1. The kinds of triangles; as, Plane, Acute, Oblique, Right, Isosceles, Scalene, Equilateral.

2. The parts of triangles; as, Sides, Base, Altitude, Vertex, Perpendicular, Hypothenuse, Legs, Median, Perimeter, Bisectors, Area.

3. Interior Angles, Exterior Angles, Vertical Angles, Base Angles.

4. Terms of Comparison; as, Homologous sides, Homologous angles, Similar triangles, Equal angles, Equal areas, Equivalent areas.

Let the learner give the derivation of Isosceles, Scalene, Equilateral, Altitude, Vertex, Perpendicular, Hypothenuse, Median or Medial, Perimeter, Area, Vertical, Homologous, and Similar.

Important Properties of Triangles.

Such properties as are deemed most important for the pupil to know will be mentioned.

1. The sum of the angles of a triangle is equal to a constant quantity, i.e., two right angles.

This theorem should be proved in two different ways. The simplest proof is by drawing a line through the vertex of the triangle parallel to the base.

From this theorem also follows: 1. Every angle of a triangle is the supplement of the sum of the other two. 2. If one side of a triangle is produced, the exterior angle is equal to the sum of the two opposite interior angles. 3. That in every triangle at least two of the angles are acute. 4. If two of the angles of a triangle are equal, they are both acute. 5. In a right-angled triangle the two acute angles are complementary.

2. Each side of a triangle is less than the sum of the other two, and greater than their difference.

3. The angle contained by two straight lines drawn from any point within a triangle to the extremities of one of the sides is greater than the angle contained by the other two sides of the triangle.

4. Conditions of Equality.

(1) When the three sides of one triangle are respectively equal to the three sides of the other.

(2) When two sides and the included angle of the one are respectively equal to the two sides and included angle of the other.

(3) When one side and two angles of the one are respectively equal to the corresponding elements of the other.

(4) Two triangles are equal when one of them has two sides, and the angle opposite to the side which is not less than the other given side respectively equal to the corresponding elements of the other triangle.

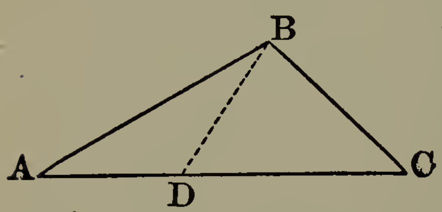
This is the ambiguous case which occurs occasionally in the construction of problems in geometry, and in the solution of certain trigonometrical problems. This case is a stumbling-block to the learner unless it is thoroughly

mastered. Perhaps the best way to impress it upon the learner's mind is to bring out what may be regarded as the exceptions to the general statement, namely, **Three Elements are enough to determine a plane triangle.** By *Elements* is meant the *three sides* and the *three angles.*

Exceptions.

1. When the three angles are given. Two unequal triangles may have their angles equal.

2. When two unequal sides and the angle opposite to the less side are given. There are two triangles that fulfill the conditions. Thus, if AB is greater than BC, and the angle BAC is given, it is evident that the triangle ABC or ABD will satisfy the conditions; hence the ambiguity. This exceptional case should be fully and thoroughly discussed before the pupil passes over it.



Conditions of Equality in Right Triangles.

1. Two right triangles are equal when the hypothenuse and a side of one are respectively equal to the hypothenuse and a side of the other.

2. When the hypothenuse and acute angle of the one are equal to the hypothenuse and acute angle of the other.

3. When a side adjacent to the right angle and the adjacent or opposite acute angle are respectively equal to the corresponding elements in another right triangle, they are equal.

Conditions of Inequality.

Here, again, the principle of contrast is employed to give effect to the teaching as well as in the method of retaining what is already learned. Both principles—equality and inequality—are used so frequently that the learner can hardly be too familiar with them. Let the learner state

each condition of equality, and then its converse; also its opposite. Very much of geometrical reasoning is carried on by the aid of equal triangles and similar triangles.

Illustrative Questions.

1. State all the conditions of equality in two plane triangles.

2. Of two right triangles.

3. Two equal lines, a and b, are joined to the line c; show that they make equal angles with c.

4. The bisector of the vertical angle of an isosceles triangle bisects the base; show that the two triangles thus formed are equal right triangles.

5. ABC is an isosceles triangle, and the angle A is twice either B or C; prove that A is a right angle.

6. If A is half of either B or C, how many degrees does it contain? How many degrees are there in the angle C?

7. The vertical angle of an isosceles triangle is 36 degrees. How many degrees in each base angle? If the sides and base be produced, determine all the exterior angles.

8. Prove that the bisectors of the equal base angles of an isosceles triangle form with the base another isosceles triangle.

Remark. Questions on a proposition may very appropriately be divided into three classes: 1. Those on the proposition itself, including agreements and differences with other propositions; 2. Numerical applications connected with the proposition; 3. Geometrical exercises. The last includes original problems. The order in the mental process appears to be: 1. Learning the new; 2. Connecting it with what is previously known; 3. Applying the knowledge to practical and theoretical problems.

Geometry is learned to be used.

The more important propositions will be dwelt upon,

that the reader may get the methods of teaching rather than a minute analysis of every theorem in an elementary treatise on geometry.

Important Theorems.

The following theorems are generally recognized as very important owing to their wide application:

1. Every point in the bisector of an angle is equally distant from the sides of the angle.

Let the pupil give the converse of this theorem.

Remark. From a discussion of these two theorems a very correct notion of what is meant by the word *Locus* in its simplest sense is obtained.

2. The three bisectors of the three angles of a plane triangle meet in a point.

3. The three perpendiculars erected at the middle points of the side of a triangle meet in the same point.

4. The three perpendiculars from the vertices of a triangle to the opposite sides meet in the same point.

5. The three medial lines of a triangle meet in the same point.

These theorems are among the most important in Book I. The fifth deserves special attention owing to its peculiar properties. The intersection of the three medials is the center of gravity of the triangle—a truth worth remembering. The geometrical demonstration is very pretty.

Questions.

1. A line is perpendicular to another line at its middle point; show that it is the *locus* of the points equally distant from the extremities of the line.

2. Is the bisector of the vertical angle of an isosceles triangle a *locus?* Illustrate.

3. If the two equal sides of an isosceles triangle decrease uniformly till the area of the triangle is zero, what kind of

a line will the vertical angle describe? What is the line with respect to the base?

4. What is the locus of a point equally distant from *two fixed* points? From two fixed parallel lines? From two intersecting straight lines?

Quadrilaterals.

Quadrilateral is from the two Latin words, *quatuor,* four, and *latus,* side. It is, therefore, any four-sided polygon. The most important simple properties of quadrilaterals relating to their angles are:

1. **The sum of its interior angles are equal to four right angles.**

2. By substituting polygon for quadrilateral, the general law is expressed thus: **The sum of all the angles of any polygon is equal to two right angles taken as many times less two as the polygon has sides.**

3. **If each side of a polygon be produced in one direction, the sum of all the exterior angles is equal to four right angles.**

The learner should observe the particular case when one or more angles are *re-entrant.*

Quadrilaterals are divided into classes as follows:

1. The Trapezium; 2. Trapezoid; 3. Parallelograms.

Parallelograms are divided into two species: 1. *Rhomboid* and *Rhombus;* 2. *Rectangle* and *Square.*

The peculiar properties of each of these figures should be learned. They may be classed under the relative directions of the sides, the nature of their angles, and the character of their diagonals.

It is from the parallelogram that the learner must get the first definite notion of finding exactly the area of a triangle.

The standard of surfaces is the square, the unit of area. Parallelograms are determined when their bases and alti-

tudes are known. Since a triangle is half a rectangle, having the same base and altitude, its area is also known. The product of two lines is a rectangle; but the product of two algebraic quantities may also be interpreted as a rectangle. The method of reasoning geometrically about magnitudes is called the *ancient* method, while the algebraic method belongs to the discoveries of Descartes.

The measurement of areas may be classed as follows:

1. Of Rectangles; 2. Of Parallelograms; 3. Of Triangles; 4. Of Trapezoids and Trapeziums; 5. Of Polygons in general.

Questions.

1. A *square* is a parallelogram whose sides are all equal and whose angles are all equal. Is this definition *defective* or *redundant?* Give reason for your answer.

2. In what respects do a trapezoid, a rhombus, and a rectangle differ? In what respects do they agree?

3. When does a parallelogram become a rectangle? A rectangle a square?

4. What figure is at once a rhombus and a rectangle? Show how this can be.

5. Which quadrilaterals have their diagonals unequal? If the diagonals of a quadrilateral be equal, what must the quadrilateral be?

6. If one angle of a parallelogram be a right angle, what must the other three angles be? Does the definition of *parallelogram* include the preceding answer? Why?

7. Is there a rule in arithmetic to find the *diagonals* of a *rectangle* if the side and end be given? What is the rule? What principle does it involve?

A good collection of exercises should conclude each book in Geometry. These exercises are more valuable, if worked by the pupils, than the theorems which are demonstrated.

The Circle.

The first element of geometry is the *right line* and the other element is the *circle*, and all constructions that can be made by these two are regarded as strictly geometrical. However, the straight line and circle are employed in trigonometry, analytical geometry, and in other departments of the more advanced mathematics.

In Elementary Geometry the more important properties of the circle may be arranged under the following subdivisions:

1. General Definitions.
2. Arcs and Chords.
3. Tangents and Secants.
4. Measurement of Angles.
5. Relative Positions of two Circles.
6. Exercises.

General Definitions.

The general definitions that must be accurately learned and retained are:

1. Circle; 2. Circumference; 3. Diameter; 4. Radius; 5. Center; 6. Arc; 7. Semi-circumference; 8. Quadrant; 9. Chord; 10. Segment; 11. Sector; 12. Tangent; 13. Point of Tangency; 14. Secant; 15. Central Angle or Angle at the Center; 16. Inscribed Angle; 17. Inscribed Segment; 18. Inscribed Triangle; 19. Inscribed Polygon; 20. Inscriptible; 21. Circumscribed; 22. Escribed; 23. Center of Symmetry; 24. Axis of Symmetry; 25. Symmetrical; 26. Conjugate Arcs; 27. Normal.

Arcs and Chords.

The principles regarded as essential which the learner should master will be enumerated under each subdivision.

1. A straight line can cut a circumference in two points only.

2. Every point at a less distance from the center of a circle than the radius is within the circumference.

3. Every point at a distance from the center equal to the radius is in the circumference.

4. Every point at a distance from the center greater than the radius is without the circumference.

5. Every diameter bisects the circumference, and also bisects the circle.

6. In equal circles, or in the same circle, equal angles intercept equal arcs on the circumference, and if the arcs are equal, the angles are equal; also in equal circles, or in the same circle, equal arcs are subtended by equal chords, and conversely.

7. The greater arc is subtended by the greater chord, if the arc is less than a semi-circumference.

8. If a diameter is perpendicular to a chord, it bisects the chord and the arcs subtended by it.

9. Equal chords of equal arcs are equally distant from the center, and of two unequal chords in the same circle or in equal circles the less is farther from the center. The converse of these propositions is also true.

10. Through three points not in the same straight line one circumference only can be drawn.

Every proposition should be proved either by the method in the text or in an independent manner by the pupil. The teacher will bear in mind that the best training is that which fits the learner to be an original reasoner, and to decide upon the validity of his own processes of reasoning. Not a few teachers of Geometry have gone so far as to exclude the text entirely from the class, and instead to assign theorems and exercises for the pupils to demonstrate according to their own devices.

Of the ten propositions under *Arcs* and *Chords,* it is evident that the fifth, sixth, seventh, eighth, ninth, and

tenth require more careful investigation than the second, third, and fourth.

At first the class need diagrams as aids, but later on these may in a great measure be dispensed with, except in rare instances, when the construction is difficult, or some obscure point is to be explained.

Figures should be seen mentally, and a little practice will enable the learner to construct such and to hold them in the mind almost as clearly as when they are outlined on the board.

Tangents and Secants.

The most elementary notion yet developed in Geometry is that of the *point,* which has position only. From the point to a line another notion or idea is involved. The two simplest conditions under which a straight line can be drawn are that *one* point of the line and its direction shall be known, or, in other words, two points determine the direction of a line. A circle is fully determined when its centre and radius are known. These two elementary notions determine the position and size of the circle. From these the circle can be drawn at once; and the next simplest condition is when three points are given not in the same straight line. Likewise the simplest problem in *Tangencies* is when a straight line is given in position and a point out of the line, to draw a circumference tangent to the given line. However, the subject of Tangencies is one of the most varied and extended in its applications within the entire range of Elementary Geometry.

As much as can be attempted in this connection is to call attention to a few of the fundamental principles underlying this interesting subject.

1. If a straight line is oblique to a radius at its extremity, it cuts the circumference; but if it is perpendicular to

the radius at its extremity, it is tangent to the circle. The converse of these propositions is true.

2. If a secant cutting the circumference is made to revolve about one of the points as a pivot, the secant becomes a tangent when the two points of intersection coincide.

This proposition shows that the tangent is a special case of the secant.

3. Two parallels intercept equal arcs on a circumference.

There are three cases:

1. When the parallels are both secants.
2. When one of the parallels is a secant, and the other is a tangent.
3. When both the parallels are tangents.

4. Two tangents to a circumference, drawn from a point without, are equal, and make equal angles with the straight line joining the point with the center of the circle.

Propositions third and fourth are the most important ones in this group. Yet *important* is used in a relative sense, and must not be interpreted as conveying the idea that the others are useless.

Measurement of Angles.

The right angle, or 90 degrees, is the natural unit of angular measurement when comparing one angle with another; but in actual practice it has been found more convenient to divide the right angle into ninety equal parts, called degrees; each degree into sixty equal parts, called minutes; and each minute into sixty equal parts, called seconds. To every angle there corresponds an arc of a circumference, and either the angle or its corresponding arc of a definite length may be taken as a medium of comparison with any other angle in the same or equal circles. Angles are compared in the same manner as other geometrical magnitudes.

The principles underlying the measurement of angles are :

1. **A central angle is measured by its intercepted arc.**

2. **A central angle is proportional to its intercepted arc.**

These principles give rise to the following theorems :

1. **An inscribed angle is measured by one half its intercepted arc.**

Case I. When one side of the angle is a diameter.

Case II. When the center is within the angle.

Case III. When the center is without the angle.

Each of these cases should be mastered in detail. The demonstrations are very simple.

In connection with these cases are associated the following deductions:

1. **All the angles inscribed in the same segment of a circle are equal.**

2. **All the angles inscribed in a semicircle are right angles.**

3. **The opposite angles of an inscribed quadrilateral are supplementary.**

4. **An angle at the center of a circle is double the angle at the circumference having the same arc.**

2. **An angle formed by a tangent and a chord is measured by one half its intercepted arc.**

3. **An angle formed by two chords intersecting within the circumference is measured by one half the sum of the arcs intercepted between its sides and between the sides of its vertical angles.**

4. **An angle formed by two secants, or by a tangent and a secant, is measured by one half the difference of their intercepted arcs.**

5. **If a quadrilateral circumscribes a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.**

Relative Positions of two Circles.

When two circles lie in the same plane, they occupy some one of the following positions with respect to each other.

1. They may be wholly external.
2. They may be tangent externally.
3. They may intersect.
4. They may be tangent internally.
5. One may lie wholly within the other.
6. They may be concentric.
7. They may be coincident when equal and concentric.

Conditions arising from Intersections.

1. If two circumferences intersect, they cut each other in two points.

2. The straight line joining the centers of two circles bisects their common chord at right angles.

3. The distance between the centers of two intersecting circles is less than the sum of their radii and greater than their difference.

Conditions arising from Tangency and Non-Intersection.

1. When two circumferences are tangent to each other externally, the distance between their centers is equal to the sum of their radii.

2. When two circumferences are tangent internally, the distance between their centers is equal to the difference of their radii.

3. When one circumference is wholly within the other, the distance between their centers is less than the difference of their radii.

If the pupil does not understand each of the important propositions enunciated, he should draw a diagram, and prove the relation required. He must study the relations of the lines, angles, arcs, and circumferences, and endeavor

to discover from what is given how to determine what is required.

Questions and Exercises Illustrating the Text.

The following questions and exercises are designed to indicate partly the method to be pursued in conducting recitations. The pupil should be strong on the side of presentation as well as in comprehension of a subject. Invention and arrangement are two desirable qualities of mind to be cultivated.

1. Four equal chords are drawn in a circumference forming a square: required the *locus* of the middle points of the chords.

2. What is the difference between a diameter and a chord? A semicircle and a segment? A secant and a tangent?

3. A diameter of a circle is 20 inches, and a chord is 10 inches; how far are they apart if the chord is parallel to the diameter?

4. A system of chords are parallel in a circle; what is the *locus* of their middle points?

5. What theorems are involved in question 4?

6. Any number of chords of a circle are drawn through a point on its circumference; what is the *locus* of their middle points? Give all the principles involved in this proposition.

7. Under what conditions will a tangent and a secant to the same circle not intersect?

8. When they do not intersect, how are they limited? Give a reason for your answer.

9. When does a secant become a chord? A tangent a point?

10. Show that tangents to a circle at the extremities of a diameter are parallel.

11. The arc of a chord is 135 degrees; what part of the whole diameter is it?

12. The diameters of two circles are 10 and 12 inches respectively: 1. If the circles are tangent externally what is the distance between their centers? 2. Tangent internally? 3. When their centers are 30 inches apart, what is the distance between their circumferences?

On the Construction of Problems.

Formerly it was the custom of American authors to give problems in one book of their geometries, and the learner was required to reproduce the work which the author had already given, instead of making his own constructions and demonstrations. Book V. of a well-known Geometry, which the writer studied first, contained all the problems,—all of which were demonstrated,—and there appeared to be no necessity for anything additional to these thirty-two problems in the text. A reaction set in, and now all American works of any value contain copious exercises. Had I to make a choice of a Geometry with no exercises, and one composed entirely of exercises and all else omitted, I would select the latter as preferable. There is an independent habit of thought derived from constructing and demonstrating problems that cannot be secured from a second-hand line of argument. All exercises are designed to test the learner's original power and skill. He fixes in his mind the given conditions. These he must hold firmly. Next he employs his constructive imagination in arranging the necessary diagram to represent not only the given conditions, but those that are required. Finally, step by step he proceeds to prove the truth or falsity of the proposition.

The object of this kind of mental discipline is to teach the learner to know truth when he finds it. It being

found, he then sets limits to the truth arrived at under the conditions imposed by the problem itself.

To construct and to demonstrate a problem involve six parts :

1. To read the problem and to remember it.
2. To draw the given parts.
3. To draw the additional parts when necessary.
4. To demonstrate it, i.e., to find the truth.
5. To find its limits.
6. To resolve it back into primary definitions.

When the pupil first begins geometry the figures are assumed to be constructed, and they are used as helps in the demonstration of principles. Later on he learns how to construct figures solely by the aid of the straight line and circumference, or the ruler and string on the blackboard, or the ruler and compasses on paper.

The successful teacher of geometry will give exercises, if the text does not contain them, under each theorem, and there should also be a large collection of well-graded exercises at the end of each book. These exercises should embrace theorems, geometrical loci, determinate problems, and algebra applied to the solution of geometrical questions. Whenever possible, geometrical principles should be traced down through algebra and arithmetic to the simplest forms under which magnitudes are investigated.

Let the fact be deeply impressed upon the learner's mind that every principle in geometry is put there for a purpose, and it is important that it should be learned, understood thoroughly, and retained. That there is no telling when it may be required in the demonstration of a theorem or in the solution of a problem, and that he is the best geometer who keeps his knowledge well in hand and can use any portion or all of it whenever the occasion demands it.

The construction of problems ought not to be postponed till the pupil or class has studied two or three books ; but

it should be carried along with the demonstration of theorems from the start. The more constructing and proving and testing the learner does, the greater the interest and the more rapid will be his progress.

Area and Equivalency.

The word "area" is used in the sense of a numerical measure, and "equivalency" is applied to magnitudes having the same size, but not the same shape. The area of a surface is the number of square units required to cover it.

Writers often use such expressions as "the rectangle of two lines, the product of two lines"—meaning thereby the product of their *numerical measures* instead of the lines themselves.

The learner should master the following principles :

1. Parallelograms having equal bases and equal altitudes are equal.

2. Every triangle is one half of a parallelogram having the same base and the same altitude.

3. Two rectangles having the same or equal bases are to each other as their altitudes, and conversely.

This truth is expressed in general by saying that any two rectangles are to each other as the products of their bases by their altitudes.

4. The area of any parallelogram is equal to the product of its base by its altitude; and the area of any triangle is equal to the product of its base by half its altitude, or half its base by its altitude.

5. The area of a trapezoid is equal to half the product of its altitude by the sum of its parallel sides.

6. The square of the sum of two lines is equivalent to the sum of their squares plus twice their rectangle.

7. The square of the difference of two lines is equivalent to the sum of their squares minus twice their rectangle.

8. The rectangle of the sum and difference of two lines is equivalent to the difference of their squares.

Propositions 6, 7, 8 should be demonstrated geometrically as well as algebraically. They are the well-known algebraic theorems, numbered I, II, and III. They afford fine illustrations of comparing the two methods of reasoning.

9. The square of the hypothenuse of a right triangle is equivalent to the sum of the squares of the other two sides.

Many demonstrations of this theorem have been given, but the 47th, in the first book of Euclid, is one of great elegance, and takes precedence over all others. The student should not only demonstrate it when the squares are described on the outer side of the triangle, but when they are described on the inner sides, or when one square is described on the outer side and two on the inner sides, or *vice versâ*.

Owing to the historic interest associated with this theorem, its great mathematical value, its wide application, and the numerous theorems dependent upon it, it is justly regarded as one of the most important theorems in geometry. The statement of the theorem as given in this connection is a special case of a general statement.

10. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of these sides and the projection of the other side upon it.

Discuss both cases. Let the learner compare the right-hand member of the equation with *Theorem II*, in algebra.

11. In an obtuse-angled triangle the square of the side opposite the obtuse angle is equivalent to the sum of the squares of the other two sides, increased by twice the product of one of these sides and the projection of the other side upon it.

Here the learner should compare the right-hand member of the equation with *Theorem I*. If AB is the side opposite

the given angle, BC and AC the other two sides, and CD is the projection, then theorems 10 and 11 are expressed thus:

$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 \mp BC \times CD$. The minus sign is used when the angle is acute, and the plus sign when it is obtuse.

These theorems have an important bearing in trigonometry.

12. **The sum of the squares of two sides of a triangle is equivalent to twice the square of the median line to the third side together with twice the square of half the third side.**

Remark. If instead of finding the equivalent of the sum of the squares of the two sides their difference be required, the expression is greatly simplified.

Nos. 10, 11, 12 show how the values may be obtained by substitution without resorting to algebraic processes. That is, the values of lines usurp the functions of unknown quantities. Frequently it becomes necessary to eliminate and to substitute unknown quantities in a geometrical demonstration, and this fact, I am inclined to believe, had much to do in laying the foundation for our modern algebraic methods.

Another and perhaps a better illustration is found in the following theorem:

13. **In any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals, increased by four times the square of the line joining the middle points of the diagonals.** However, this proposition is little used in geometry, but it is good practice for the learner to make the reduction. If the length of the line joining the middle points of the diagonals is zero, a new phase of the problem is presented, which the learner will readily recognize. These unexpected turns add great interest to mathematical investigations.

14. To find the area of a triangle when its three sides are given.

This problem is given as a rule in the Common School Arithmetics without a demonstration, but the pupil will find a demonstration in geometry or trigonometry which he should work out. After solving it, he will then understand why it is not given in the arithmetics. As a general thing, the student of mathematics should solve this problem every time he feels himself in doubt about it.

Proportionality and Similarity.

Definitions are to be handled as sharp cutting instruments. They are the tools of thought in mathematical studies. The terms of frequent occurrence in this chapter to be learned are: 1. Proportionality; 2. Similarity; 3. Internal Segments; 4. External Segments; 5. Similar Polygons; 6. Homologous Points, Lines, or Angles; 7. Ratio of Similitude; 8. Mean Proportional; 9. Third Proportional; 10. Fourth Proportional; 11. Extreme and Mean Ratio; 12. Harmonically; 13. Harmonic Points.

The essential propositions to be demonstrated are:

1. A parallel to one side of a triangle, intersecting the other two sides, divides these sides proportionally.

What relation exists between the sides and their *internal segments?* If the sides be produced, what relation exists between the sides and their *external semgents?*

If two lines are intersected by any number of parallels, how are the two lines divided? If a straight line divides two sides of a triangle into proportional internal segments, is the straight line parallel to the third side? Why? If proportional to the external segments? Why?

2. In any triangle the bisector of an angle divides the opposite side into segments proportional to the adjacent sides.

If the angle be exterior, what modification must be

made? Draw a figure and demonstrate. Give numerical values to the three sides of each triangle, and then find both the internal and the external segments. Is it necessary to know the angle in degrees in order to find the segments. Illustrate your answer. Give the converse of these propositions, and demonstrate the same.

Conditions of Similarity.

3. Triangles are similar when they are mutually equiangular.

4. Triangles are similar when an angle in the one is equal to an angle in the other, and the sides including these angles are proportional.

5. Triangles are similar when their homologous sides are proportional.

6. Triangles having their sides respectively parallel or perpendicular are similar.

The conditions of equality and similarity of triangles should now be compared in every respect, and the distinctions also be made between each of these and the equivalency of two triangles.

7. The areas of two similar triangles are to each other as the squares of their homologous sides.

The more general proposition, however, is, that the areas of two similar polygons are to each other as the squares of any two homologous lines. Frequently in the solution of a numerical or algebraic problem it becomes necessary to find a homologous line in one polygon when the two areas and the homologous line of the other polygon are given. In that case the homologous sides of two similar polygons have the same ratio as the square roots of their areas. Many arithmetical problems are solved by this principle.

Illustrative Exercises.

1. A field is 160 by 60 rods: what will be the length and breadth of a tract of land 4 times as large?

2. The homologous sides of two similar triangles are to each other as 6 to 10, and the sum of their areas is 832 square inches: find the area of each triangle.

8. When two chords intersect within a circumference, their segments are reciprocally proportional.

This proposition is more generally expressed thus: If through a fixed point within a circumference a chord is drawn, the product of the two segments is *a constant* in whatever direction the chord may be drawn.

9. When two secants intersect within a circumference, the whole secants and their external segments are reciprocally proportional.

The two triangles formed are mutually equiangular and similar. If one of the secants becomes a tangent, then the tangent is a *mean proportional* between the whole secant and its external segment. Also if a secant is drawn through a fixed point within a circumference, the product of the whole secant by its exterior segment is a *constant* in whatever direction the secant may be drawn.

Propositions 6 and 7 are two of the most important in elementary geometry, and should be thoroughly mastered by the pupil.

Problems in Equivalent Areas.

1. To construct a square equivalent to two given squares. This problem may be extended so as to embrace any number of squares. Let the pupil discover the method.

2. To construct a square equivalent to the difference of two given squares.

3. To construct a square equivalent to a given parallelogram.

4. Upon a given straight line, to construct a rectangle equivalent to a given rectangle.

5. To construct a triangle equivalent to a given polygon.

6. To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.

Three or four recitations should be given to these problems, or at least as much as is necessary for the class to master them. All are important.

Other problems following directly after these in treatises on geometry should next be mastered in detail.

Regular Polygons. Measurement of the Circle.

Definitions to be Learned.

1. **A Regular Polygon.** Give examples.

2. **Center of a Regular Polygon.** When *Inscribed,* when *Circumscribed.* What do these two words mean? From what are they derived? To what conjugation does the word *scribo* belong?

3. **Perimeter of a Regular Polygon.** Analyze the word "perimeter." From what language is it derived?

4. Define *Apothem.* What does it mean in Geometry? *Similar Arcs? Similar Sectors?*

5. **Radius of a Regular Polygon.**

The discussion of Regular Polygons leads ultimately to the finding of the circumference of the circle. The steps are gradual, beginning with the equilateral triangle inscribed in a circle, then a square, and so on to a regular polygon of a very great number of sides. In dealing with these geometrical figures, several things have to be kept constantly in mind: the number of sides of the polygon, the angles at the centre and how obtained, the ratio between the radius or diameter and the perimeter of the polygon, its area, etc.

The principal propositions of Elementary Geometry touching polygons will now be stated.

1. **If a circumference be divided into an equal number of arcs, and a chord be drawn in each arc, these chords will form the sides of a regular polygon.**

This proposition depends upon the equal divisions of the circumference.

2. **A circle may be circumscribed about, or inscribed within, any regular polygon.**

It is here assumed that these things can be done, but the learner should show how they are done.

3. **Regular polygons of the same number of sides are similar.** Show that they are composed of similar triangles.

4. **The perimeters of two similar regular polygons are to each other as the radii of their circumscribed or of their inscribed circles; and their areas are to each other as the squares of their radii, or of any two homologous lines.**

Of what special proposition is this a more general statement? Is there a more gēneral statement yet? Why?

5. **The area of a regular polygon is equal to one half the product of its perimeter by its apothem; or its perimeter by half its apothem.**

Upon what proposition does this depend? This proposition is stated in another form, namely, that the area of a regular polygon is equal to its perimeter multiplied by half the radius of the inscribed circle.

To be done.

At this stage in the learner's progress, he should—

1. Inscribe a square in a given circle.
2. Inscribe a regular hexagon in a given circle.
3. Inscribe an equilateral triangle in a given circle.
4. Inscribe a regular decagon in a given circle.
5. Inscribe a regular polygon of fifteen sides.

The 4th and 5th are very pretty problems which should be constructed without fail.

To divide a circle into 6, 12, 24, 48, etc., equal parts is another but simpler phase of inscribing a certain class of regular polygons in a circle; also a circle can be easily divided into 5, 10, 20, etc., parts.

Approximating the Circumference of a Circle.

With what has preceded, it is now necessary for the learner to make still further advances toward determining the circumference of a circle.

It is easily shown that an *arc of a circle is less* than any line which envelops it and has the same extremities. As a corollary to this, the circumference of a circle must *be less* than the perimeter of any polygon circumscribed about it. So far two steps have been taken which the learner must carefully note.

Again, if the number of sides of a regular polygon inscribed in a circle be increased indefinitely, the apothem of the polygon will approach to the radius of the circle as its limit. Finally, the circumference of the circle is the *limit* to which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is increased indefinitely; consequently the area of the circle is the limit of the areas of these polygons.

The learner should not be content, however, with what has been thus briefly sketched. He should now proceed to find the length of the circle itself in terms of a right line. This is called the *rectification of the curve.*

The first approximation is made under the following conditions: Given the perimeters of a regular inscribed and a similar circumscribed polygon, to find the perimeters of the regular inscribed and similar circumscribed polygons of double the number of sides.

Using the common notation, the two equations

$$(1) \qquad p' = \sqrt{p \times P} \quad \text{and}$$

$$(2) \qquad P' = \frac{2p \times P}{P + P} \quad \text{are obtained.}$$

Here p and P represent the perimeters of a regular inscribed and regular circumscribed polygon of the same number of sides, and p' and P' the perimeters of the regular inscribed and circumscribed polygons of double the number of sides. From equations (1) and (2) the learner should compute some of the values given in the tables of approximation.

Instead of using the perimeters of the regular inscribed and circumscribed polygons, the *radius* and *apothem* of a regular polygon may be employed to compute the *radius* and *apothem* of the isoperimetric polygon of double the number of sides.

Since the ratio of the circumference of a circle to its diameter is *constant*, this constant is represented by π, so that for any circle whose diameter is $2R$, and circumference C, we have the relation

$$\frac{C}{2R} = \pi, \qquad \text{or} \qquad C = 2\pi R.$$

Now if the circle be conceived as made up of an infinite number of triangles, all having their vertices at the center of the circle, then the circumference of the circle will form the sum of all the bases of these triangles, and the radius of the circle will be the altitude of these triangles; that is, the difference between the sides of these isosceles triangles and their altitudes is infinitely small. Hence the area of the circle will evidently be

$$2\pi R \times \frac{R}{2} = \pi R^2 = \text{Area}.$$

Exercises on Polygons.

There are certain relations existing among the sides, radius, apothem, and area of the regular inscribed polygons which the learner should determine by a combination of his algebraic and geometrical knowledge.

Let the learner prove the following, where R = radius of the regular inscribed polygon, r = apothem, a = one side, A = anterior angle, c = angle at center:

1. In a regular inscribed triangle, $2 = R\sqrt{3}$, $r = \frac{1}{2}R$, $A = 60°$, $C = 120°$.

2. In an inscribed square, $2 = R\sqrt{2}$, $r = \frac{1}{2}R\sqrt{2}$, $A = 90°$, $C = 90°$.

3. In a regular inscribed hexagon, $a = R$, $r = \frac{1}{2}R\sqrt{3}$, $A = 120°$, $C = 60°$.

4. In a regular inscribed decagon, $a = R\dfrac{\sqrt{5}-1}{2}$, $r = \frac{1}{4}R\sqrt{10+2\sqrt{5}}$, $A = 144°$, $c = 36°$.

These exercises may very properly be extended so as to include the octagon, dodecagon, and other regular polygons.

Maxima and Minima of Plane Figures.

Definitions.

1. Geometrical forms, under restricted conditions, may have a **Maximum** or **minimum** form.

2. **A Maximum Figure** is the greatest one of its kind.

3. **A Minimum Figure** is the least one of its kind.

4. **Isoperimetrical Figures** are those which have equal perimeters.

The subject of maxima and minima is very briefly treated in American works. Owing to its extensive application not only in the solution of pure geometrical problems, but in the calculus also, it deserves much more than a

mere passing notice. It was cultivated by the ancient geometers with marked success, and more recently by the English with much greater diligence than by those of the Continent.

A few of the simple propositions only will be referred to in this connection,—enough, however, to give the learner a glimpse of this interesting subject.

Propositions.

1. The shortest distance from a given point outside of a given line is the perpendicular to the line.

2. The greatest straight line that can be drawn in a circle is a diameter.

3. The shortest straight line that can be drawn through a fixed point within a circumference is the chord at right angles to the line joining the center and the fixed point.

4. The sum of the lines drawn from two fixed points on the same side of a given line to a point in that line is the least when the lines make equal angles with the fixed line.

This is one of the most beautiful theorems in geometry.

5. Of all triangles formed with two given sides, the greatest is that when these two given sides make a right angle.

6. Of all triangles having the same base and equivalent areas, the isosceles triangle has the least perimeter.

7. The sum of two adjacent sides of a rectangle being constant, the area is the greatest when the sides are equal.

8. Among isoperimetric triangles with a constant base, the greatest is isosceles.

How is the proposition modified if the base is *not constant?* Why?

9. Of all plane figures containing the same area, the circle has the least perimeter.

10. Of all equivalent polygons of the same number of sides, the one of the least perimeter is regular.

11. Of all isoperimetrical polygons of the same number of sides, the one which is regular has the greatest area.

Exercises.

1. With 360 panels of fence how can a farmer enclose the largest amount of ground? Illustrate your answer.
2. Divide the number 36 into such parts that their products will be the greatest possible. What are the parts?
3. What *three plane figures only* can be used to cover completely the space about a point in a plane? Why?
4. Find a point on a semi-circumference that the *sum* of the distances to the ends of diameter is the *greatest possible.*
5. How should the roof of a barn made of two slant sides be pitched so as to hold the most hay?

GEOMETRY OF SPACE.

Plane Geometry is restricted to points, lines, angles, and surfaces lying in a plane. It is comprised within two dimensions, length and breadth. Geometry of Space includes an additional dimension—*thickness.* A clear and comprehensive knowledge of plane figures lays the foundation for a correct understanding of Spacial Geometry. The geometry of the *point* is position; of the line, *extent;* of the plane figure, length and breadth; and of the solid, length, breadth, and thickness.

All plane figures can be represented by suitable diagrams on a sheet of paper, slate, or blackboard; but in Geometry of Space this is impossible. Whenever the learner fails to see *mentally* the figure he wishes to discuss, he should use sticks, wires, threads, or pins to represent lines, and stiff card-board or heavy pieces of paper to represent surfaces and solids.

In general, it is better to use these or similar helps in the demonstration of all difficult propositions.

Definitions and General Principles.

The learner must now familiarize himself with the mental process of passing from a plane into space, and to see spacial relations with as great facility as he has hitherto studied them in a plane. Here, however, he is obliged to depend upon *definitions* and *general principles* in order to make the most rapid progress. The following terms and principles in connection with the axioms and postulates constitute the foundation of Geometry of Space, and so much importance is attached to them that progress is impossible until they are clearly and critically apprehended.

To be Mastered.

1. Plane. 2. When is a plane determined? 3. How is a plane determined? First? Second? Third? Fourth? Illustrate how a plane is determined by a *line* and a *point*. How is it *not* determined by a *point* and a *line?* Illustrate how three points may or may not determine a plane. Under what condition will two straight lines determine a plane? Will two straight lines that do not intersect determine a plane? Illustrate. 4. When is a straight line perpendicular to a plane? 5. When oblique to a plane? 6. When is a line parallel to a plane? 7. When are two planes parallel? 8. What is the projection of a point on a plane? 9. What is the projection of a line on a plane? 10. Why will a straight line not determine the position of a plane? 11. What is the *inclination* of a line to a plane?

Elementary Propositions to be Proved.

1. The intersection of two planes is a straight line. Why?

2. But one perpendicular can be drawn from a point out of a plane to that plane.

3. All perpendiculars drawn to a given point in a line lie in one plane.

4. A straight line perpendicular to two other straight lines at their point of intersection is also perpendicular to the plane in which the two straight lines lie.

5. If from a point out of a plane lines be drawn meeting the plane at equal distances from the foot of the perpendicular, these lines are equal, and lines meeting the plane at the greater distance from this foot are longer.

6. A straight line parallel to a line in a plane is also parallel to the plane.

7. If a line is perpendicular to a plane, every line parallel to the given line is perpendicular to the plane.

8. Two planes perpendicular to the same line are parallel.

9. When two angles not in the same plane have their sides parallel and extend in the same direction, the angles are equal and their planes are parallel.

10. If two straight lines are cut by three or more parallel planes, the corresponding segments are in proportion.

Diedral Angles.—Angle of a Line and Plane.

Definitions.

1. When two planes meet and are terminated by their common intersection, they form a *diedral angle.* The planes are its *faces,* and the intersection is its *edge.* An open book represents a *diedral* angle. The two pages are the planes; and where the leaves join is their intersection. A diedral angle is measured by the difference of the opening between the two leaves, or by a straight line in each face drawn perpendicularly to the same point in the edge. An angle thus formed is called the *plane angle* of the diedral.

2. The *magnitude* of a diedral depends upon the difference of the divergence between its two faces, i.e. from 0° to 360°. When the two faces form *one plane,* the angle is 0°, or 180°, or 360°. They form two angles, an inside angle and an outside angle. Their sum is constant, and equals 360° when one complete revolution is made. Hence a *diedral* angle may be any angular quantity. A *diedral* angle may be equal to any sum or to any difference of angles, or it may bear any ratio to any other angle.

3. *Diedrals* are *acute, obtuse, supplementary,* etc.

4. *All right diedrals are equal.* The learner will observe the agreements in 3 and 4 with those in regard to plane angles formed by intersecting straight lines.

5. *When two planes pierce each other, the opposite or vertical angles are equal.* How does this correspond to a similar proposition in Plane Geometry?

Propositions.

1. **Two diedral angles have the same ratio as their plane angles.**

2. **When a straight line is perpendicular to a plane, every plane which passes through the line will be perpendicular to that plane.** Why? Illustrate.

3. **When two planes are perpendicular to each other, if a straight line in one is perpendicular to their intersection, it will be perpendicular to the other plane also.**

4. **Through any straight line a plane can be passed perpendicular to any given plane.**

5. **If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.**

6. **Every point of a plane which bisects a diedral angle is equally distant from its two faces. The bisecting plane is the locus of all the points equally distant from the two**

given planes. How is this statement modified if the bisecting plane pierces the other two?

7. **The projection of a point upon a plane is the foot of the perpendicular let fall from the given point upon the plane.** Consequently, the projection of any line on a plane is the projections of all the points of the line upon the plane.

8. **The acute angle that a straight line makes with its own projection upon a plane is the least angle which it makes with any line of the plane.** This angle is called the *inclination of the line to the plane,* or the *angle formed by the line and the plane.*

Polyedral Angles.

1. **When three or more planes meet in a common point, they form a polyedral angle.** The point where they meet is the *vertex.* The intersections of the planes are the *edges,* the portions of the planes between the edges are the *faces,* and the angles formed by the edges are *face-angles.*

2. The simplest form of the polyedral is the *triedral* angle having three faces.

3. The **magnitude** of a polyedral angle depends solely upon the relative position of the faces.

4. Two polyedrals are *equal* when they can be made to coincide, or can be imagined as coinciding.

5. A polyedral is *convex* when a plane cutting all its faces forms a *convex polygon.*

6. **Symmetrical polyedrals** are those whose elements are respectively equal, but are arranged in a reverse order. If the edges of a polyedral be produced beyond the vertex, they will form the edges of a new polyedral which is symmetrical with the first. These two polyedrals are *equal by symmetry,* but they cannot be placed so that their faces will coincide.

Remark. The learner will obtain a good idea of symmetry by taking three or more pieces of card-board and putting them together so that they form symmetrical polydedral angles. A material illustration will greatly assist him in understanding thoroughly this special department of Geometry.

Propositions.

1. The sum of any two face-angles of a triedral angle is greater than the third face-angle. If the sum of the two face-angles become equal to the third face-angle, what will the triedral be? Why?

2. The sum of the face-angles of any convex polyedral angle is less than four right angles.

The usual demonstration given of this proposition is not always satisfactory to the learner. He sees the truthfulness of the proposition, but the proof is slightly obscure.

To make it plain at the beginning, take a point above the plane of the paper and imagine three lines drawn from the given point to three other points on the paper. These three lines will be the three edges of a triedral angle. Now, let the vertex of this triedral be depressed towards the plane of the paper, and the limit of the face-angles will be 360°. When the triedral becomes a plane triangle this limit is reached. Again, suppose the vertex to be raised *indefinitely* above the plane of the paper, and at the moment when the edges of the triedral are parallel each vertical angle is 0°. Hence the limits are 0° and 360.°

The learner is ready now to study with profit the usual demonstration. In the demonstration it is better to use all the *inequations,* rather than to use one or two and then infer the others, and finally jump at the conclusion.

3. Two triedral angles are equal or symmetrical if the three face-angles of one are respectively equal to the three face-angles of the others.

4. In every triedral the sum of the diedrals is greater than two right angles and less than six right angles.

In this proposition the supplementary triedral may be considered in connection with the triedral; or better still, a pasteboard triedral can be used, and from a point within it let perpendiculars be drawn to each face; then the conditions expressed in the proposition can be easily and simply demonstrated.

The importance of thorough work here cannot be overestimated, on account of its bearings on spherical trigonometry. The roots of that special department reach down deep into this proposition.

Exercises.

1. The three planes which bisect the diedral angles of a triedral meet in the same straight line.

2. What is the locus of all the points at a given distance from a given plane?

3. Find the locus of all the points any one of which is equally distant from three given points.

4. What is the locus of all the points any one of which is equally distant from three given planes?

5. If two convex polyedrals have the same number of faces, the sum of the diedrals of one is within four right angles of the sum of the diedrals of the other.

6. If perpendiculars are drawn from a point within a diedral angle perpendicular to its faces, the angle included between the perpendiculars is the supplement of the diedral angle.

7. If three perpendiculars are respectively drawn from a point within to the faces of a triedral, what are the inferior or superior limits of the three supplementary diedral angles?

8. Find the locus of the points any one of which is equally distant from two given points, and also equally dis-

tant from two given straight lines which lie in the same plane.

9. Find the locus of the points which are equally distant from two given points, and also from two given lines.

10. Find the locus of the points which are equally distant from two given straight lines in the same plane, and also from two given planes.

11. In how many different ways may three different planes intersect? Illustrate by diagrams.

Polyedrons.

The study of diedrals and triedrals prepares the way for that of *Polyedrons.* Although the little child studies many of the geometrical forms in the lower grades, yet his knowledge of them at best is very inadequate compared to the scientific accuracy which is now demanded. But here the learner needs material forms to aid him in his investigations.

Before he can see pure geometrical forms, he must obtain clear and exact ideas from the material illustrations repre senting the pure forms. Let him, therefore, construct each form as he learns a definition of it. The appeal made through sight and touch greatly re-enforce the imagination in constructing an adequate conception of the form itself. Ideas obtained in this way are retained longer than those fleeting impressions which have no fixed abode in the mind. Here, again, knowledge rests on definitions, generalized and comprehended.

The following terms should be defined, learned, and illustrated: 1. Polyedrons; 2. Faces; 3. Edges; 4. Vertices; 5. Diagonal; 6. Tetraedron; 7. Hexaedron; 8. Octaedron; 9. Dodecaedron; 10. Icosaedron; 11. Convex; 12. Volume; 13. Measure; 14. Unit; 15. Equivalent; 16. Equal; 17. Prism; 18. Surface; 19. Altitude; 20. Triangular; 21. Quadrangular; 22. Right; 23. Oblique;

24. Regular; 25. Parallelopiped; 26. Cube; 27. Pyramid; 28. Slant; 29. Truncated; 30. Frustum.

Propositions.

1. Parallel sections of a prism are equal polygons.

2. The lateral surface of a right prism is equal to the perimeter multiplied by the base.

3. The four diagonals of a parallelopiped bisect each other.

4. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.

5. Two prisms are equal if three faces, including a triedral angle of the one, are respectively equal to three faces similarly arranged, including a triedral of the other.

6. An oblique prism is equivalent to a right prism whose bases are equal to right sections of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.

7. Two rectangular parallelopipeds having equal bases are to each other as their altitudes.

8. Two rectangular parallelopipeds are to each other as the products of their three dimensions.

9. The volume of a rectangular parallelopiped is equal to the product of its length, breadth, and height.

10. The volume of any parallelopiped is equal to the product of its base by its altitude; also, the volume of any prism is equal to the product of its base by its altitude.

11. When a pyramid is cut by a plane parallel to its base, the lateral edges and the altitude are divided proportionally, and the section is a polygon similar to the base.

12. Two triangular pyramids having equal altitudes and equivalent bases are equivalent.

13. The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one-half its slant height.

14. A triangular pyramid is one-third of a triangular prism of the same base and altitude.

Remark. The learner should take a potato or an apple, and verify the 14th proposition by cutting a triangular prism from it, and cutting the prism into three equivalent triangular pyramids. The importance of this proposition cannot be overestimated, since it is the basis for ascertaining the volume of the cone and sphere as well as that of the pyramid.

15. The volume of any pyramid is equal to one-third of the product of its base by its altitude.

The volume of a frustum of a pyramid is equal to one-third of the product of its altitude by the sum of its lower base, its upper base, and the mean proportions between its bases.

This is most readily demonstrated by completing the pyramid, and then finding the difference between the whole pyramid and the added one.

17. Similar polyedrons are to each other as the cubes of their homologous edges. This proposition may be stated more generally thus: **Two similar polyedrons are to each other as the cubes of any two homologous lines.**

Exercises.

1. There is a point equally distant from the four vertices of a tetraedron.

2. There is a point equally distant from the four faces of a tetraedron.

3. The six planes that bisect the diedrals of a tetraedron meet in a point.

4. A pyramid can be divided into the same number of tetraedrons as its base can be divided into triangles.

5. The four straight lines that join the vertices of a tetraedron with the intersections of the medians of the opposite faces meet at a common point, and are in the ratio of 3 to 1.

6. The distance of the center of a parallelopiped from any plane is equal to one-eighth of the sum of the distances of its eight vertices from the same plane.

7. Cut a cube by a plane so that the section shall be a regular hexagon.

8. The base of a regular pyramid is a hexagon each of whose sides is 3 meters in length, and its convex surface is 6 times the area of its base: required its height.

9. What is the volume of a rectangular parallelopiped whose length and width are as 3 to 2, and whose surface is 208 inches?

10. Prove that the base of a pyramid is less than its lateral surface.

11. If the three dimensions of a rectangular parallelopiped be denoted by a, b, c, what would be the expressions for the surface, volume, and diagonals?

12. The edges of a regular tetraedron are 6 inches each: required the length of a straight line drawn from a vertex to the middle point of the opposite face.

13. The entire surface of a regular tetraedron is to its volume as 4 to 5: what is the length of one of its edges?

14. If a wheat-bin is 20 feet long, 12 feet wide, and 10 feet deep, what will be the dimensions of a similar bin which holds ten times as much grain?

The Regular Polyedrons.

This discussion depends upon the definition of the word *Regular* as it is applied to Polyedrons. It has been shown in Elementary Geometry that a plane may be covered about a point by six equilateral triangles, four squares, or by three regular hexagons, and that any other case is impossible.

Consequently, the problem of the Regular Polyedrons is to determine the number of regular convex polyedrons that can be constructed so as to enclose a definite form.

To surround a limited space, a convex polyedral angle must have three faces, and the sum of the three face-angles must be less than 360°; for if the sum of the face-angles be 360°, then the solid angle becomes a plane surface.

1. By using one of the face-angles of an equilateral triangle as a measuring unit, we find that the following combinations of convex polyedral angles may be formed, namely: of three, four, or five equilateral triangles. A fourth case is impossible, since six equilateral triangles will reduce the polyedral angle to a plane.

2. It is also evident that a convex polyedral angle may be formed with three squares, but not with four. One case is therefore possible.

3. Since a face-angle of a regular pentagon contains 108°, a convex polyedral angle may be formed by combining three regular pentagons. Four such angles are greater than 360°, hence one case is possible.

4. Three angles of a regular hexagon are equal to 360°, and of a regular heptagon greater than 360°; hence only five regular convex polyedrons are possible. They are named, from the number of faces composing each polyedral angle, as follows: 1. *Tetraedron;* 2. *Hexaedron, or Cube;* 3. *Octaedron;* 4. *Dodecaedron;* 5. *Icosaedron.*

Before constructing these polyedrons, the learner should make models out of cardboard. They should be cut out entire, and the lines separating adjacent polygons should be cut half through the cardboard, and the edges turned to the proper form, when they can be sewed or glued. The tetraedron is composed of four regular equilateral triangles; the cube, of six squares; the dodecaedron, of twelve regular pentagons; and the icosaedron, of twenty regular equilateral triangles.

The next step is for the learner to construct each of these polyedrons, beginning with them in the order enumerated. To construct a regular polyedron, one of the faces,

or an edge, should be known; then all the other dimensions can be readily determined.

Euler's Theorem on Polyedrons.

Leonard Euler, the distinguished mathematician, discovered the following beautiful theorem which bears his name:

"In any polyedron, the number of its edges increased by two is equal to the number of its vertices increased by the number of its faces."

The theorem may be expressed thus: $E + 2 = V + F$, where E denotes the number of edges; V, the vertices; and F, the faces.

This equation is easily demonstrated by taking any one of the five regular polyedrons and removing one face, and then noting the number of edges, of vertices, and of faces remaining.

Beginning with one face of an equilateral triangle, say, the edges equal the vertices; next annexing a second face, we have an edge and two vertices in common with the first.

Or, for 2 faces, $E = V + 1$;
for 3 faces, $E = V + 2$;
for 4 faces, $E = V + 3$; and
for N faces, $E = V + N - 1$.
F faces, $E = V + F - 2$.
Or, $E + 2 = V + F$,

which proves the theorem.

Exercises.

1. The sum of all the face-angles of any polyedron is equal to four right angles taken as many times, less two, as the polyedron has vertices.

2. The polyedron which has for its vertices the centers of the four faces of a regular tetraedron is also a tetraedron

3. What ratio do the tetraedrons in No. 2 bear to each other, if the edge of the circumscribing one be 10 inches?

4. An octaedron is inscribed in a cube, the vertices being at the centers of the faces of the cube: what part of the volume of the cube is the octaedron?

The Three Round Bodies.

Elementary Geometry treats of only three bodies bounded by curved surfaces. These are the *cylinder*, the *cone*, and the *sphere*. It is more logical to treat each of these bodies in detail than to present a few of the properties of one and then begin with another, and so on.

The Cylinder.

Definitions.

1. The cylinder. 2. How is it generated? 3. Literal meaning. 4. Surface. 5. Bases. 6. Altitude. 7. Generatrix. 8. Directrix. 9. Right cylinder. 10. Cylinder of Revolution. 11. Radius. 12. Circumscribed. 13. Inscribed. 14. Elements. 15. Volume. 16. A material cylinder. 17. Similar cylinders.

Principles and Propositions.

1. Every section of a cylinder embracing two elements is a parallelogram. Illustrate.

2. Any section of a right cylinder parallel to its lower base is a circle.

3. The convex surface of a cylinder is equal to the perimeter of the base multiplied by its altitude.

How must this statement be modified if the area of the two bases be included? What formula represents the convex surface of a cylinder? The entire surface?

4. The volume of a cylinder is equal to the product of

its base by its altitude. What does $V = \pi r^2 \times a = \pi r^2 a$ mean?

The Cone.

Definitions.

1. Definition of Cone. 2. How generated. 3. The generatrix. 4. The directrix. 5. The vertex. 6. The base. 7. Altitude. 8. Surface. 9. Element. 10. Nappes. 11. Right. 12. Oblique. 13. Circular. 14. Volume or Solidity. 15. Similar cones. 16. Truncated cone. 17. Frustum. 18. Axis. 19. Slant. 20. Inscribed. 21. Circumscribed.

Properties.

1. Any section made by passing a plane through the vertex of a cone and cutting its base is a triangle.

2. If a plane cut a right cone parallel to its base, the section is a circle.

3. The convex surface of a right cone is equal to the circumference of its base multiplied by its slant height.

What is the difference between the convex surface of a cone and its entire surface?

4. The convex surface of the frustum of a right cone is equal to the sum of its bases multiplied by half its slant height.

5. The volume of a cone is equal to the area of its base multiplied by one third of its altitude.

6. The volume of the frustum of a cone is equal to one third of the product of its altitude by the sum of its lower base, its upper base, and a mean proportional between the two bases.

Exercises.

1. The slant height of a right cone is twice the diameter of its base: what is the ratio of the area of the base to the lateral surface?

2. How many yards of domestic $\frac{3}{4}$ of a yard wide will be required to make a conical tent 15 feet in diameter and 10 feet high?

3. Given the total surface S of a right cone, and the radius r of the base, to find the volume V.

4. Given the total surface S, and the lateral surface S', to find the volume V.

5. The altitude of a right cone is 6 meters, and the diameter of its base 4 meters: required its slant height, lateral surface, area of base, and volume.

6. The lateral area A of a right cone being given, what is the relation between its altitude a and the diameter D of its base?

7. Two similar cones are to each other as the cubes of their like dimensions.

The Sphere.

It would be more logical to treat of the surface and volume of the sphere before passing to the consideration of spherical triangles and their properties. Yet this does not follow the usual order of presentation in most American treatises.

Definitions.

1. Sphere. 2. How generated. 3. Meaning of the word "sphere." 4. Diameter. 5. Radius. 6. Center. 7. Surface. 8. Volume. 9. Great circle. 10. Small circle. 11. Tangent plane. 12. Point of contact. 13. Regular semi-perimeter. 14. Regular semi-polygon. 15. Inscribed cylinder. 16. Circumscribed cone. 17. Inscribed cone. 18. Circumscribed cone. 19. Tangent spheres. 20. Concentric spheres.

Propositions.

1. Any plane section of a sphere is a circle.

2. The largest possible section is that which passes through the center of the sphere.

3. Two great circles of a sphere bisect each other.

4. The intersections of two circumferences of great circles on the surface of a sphere are distant 180°.

5. The surface described by the revolution of a regular semi-polygon about its axis is equal to the product of the axis by the circumference of the inscribed circle.

6. The area of the surface of a sphere is equal to the product of the diameter by the circumference of a great circle.

7. The volume of a sphere is equal to one third of the product of the surface by the radius.

8. The surface of a sphere is equal to the convex surface of the circumscribing cylinder.

9. The volume of the sphere is two thirds of the circumscribing cylinder.

10. If a cone and a cylinder have for their respective bases a great circle of a sphere, and their altitudes are equal to the diameter of the sphere, their volumes are to each other as 1 : 2 : 3.

Spherical Surfaces and Volumes.

The simplest form of spherical surfaces is the spherical angle. The geometrical figures that can be made on the surface of a sphere are as various as those in plane geometry. They are, however, as easily understood if properly approached. Two methods are adopted in presenting this subject: one begins with the figures on the surface of the sphere; the other conceives planes as intersecting at the center of the sphere and passing outward to the spherical surface which they cut. The latter method is the more satisfactory. The different kinds of spherical surfaces and

volumes may be grouped under the following heads, namely: 1. Spherical angles. 2. Spherical triangles. 3. Spherical polygons. 4. Line. 5. Wedge. 6. Zone. 7. Segments. 8. Sector. 9. Pyramid. 10. Hemisphere. In teaching Spherical Geometry the learner should be led to observe the analogies and contrasts between lines made on a plane and those made on the surface of a sphere. As illustrations: distances in plane geometry are measured along a straight line, while in spherical geometry distances are measured on the arc of a great circle. In plane geometry the sum of the three angles of any plane triangle is always *constant*—equal to 180°; but in spherical geometry the sum of the three angles is always greater than two right angles and less than six right angles. The first conception the learner obtains of a spherical angle should be clear and precise. Its measurement should be definitely fixed in his mind. To assist him, a blackboard sphere which revolves as easily as possible on its axis should be used to draw the triangles and their polars upon. Before demonstrating any of the properties of spherical triangles, the learner should devote some time to drawing triangles, their polars, and symmetrical triangles. An ordinary paper globe may be used advantageously in the absence of a regular blackboard globe.

Propositions.

1. A spherical angle is measured by the diedral angle which is included by the planes of its sides.

2. Every spherical triangle corresponds to a triedral angle at the center of a sphere, having its six parts equal to the six parts of the spherical triangle.

3. Any triedral angle may be represented by a triangle on the surface of a sphere. The vertex of the triedral is at the center of the sphere.

Consequently: (1) If two sides of a spherical triangle are

equal, the opposite angles are equal. (2) The greater side is opposite the greater angle, and conversely. (3) The sum of any two sides is greater than the third side. (4) Two spherical triangles on the same sphere are equal when their sides are equal and equally arranged. (5) The sum of the three sides of a spherical triangle is less than 360°. (6) Two opposite spherical triangles correspond to two opposite symmetrical triedral angles. (7) If two sides of a spherical triangle are quadrants, the angles opposite these sides are right angles. (8) Three great circles divide the surface of a sphere into eight spherical triangles, as three intersecting planes divide the space about a point into eight parts; hence every spherical triangle has seven others whose sides and angles are either equal or supplementary to those of the given triangle. (9) If the planes of three great circles are perpendicular to one another, they form eight equal tri-rectangular triedrals at the center of the sphere, and the eight corresponding triangles have all their angles right angles, and all their sides are quadrants. Such triangles are called *quadrantal triangles.* Lest the learner become confused in regard to spherical triangles and their triedrals, he needs to be cautioned that a spherical triangle has a definite area, and that its perimeter has a definite length, both depending upon the diameter of the sphere, while all the elements of a triedral are angular quantities and answer as well for a small sphere as a large one.

4. The radii being equal, two spherical triangles are equal when—(1) The three sides are equal. (2) The three angles are equal. (3) Two sides and the included angle are equal. (4) One side and the adjacent angles are equal. (5) In all cases of equal elements, when the arrangement is the same, the triangles are equal; but when reversed, they are symmetrical.

Polar Triangles.

If a perpendicular be erected to every face-angle at the vertex of a triedral angle, these lines form the edge of a supplementary triedral; and if the vertex be at the center of a sphere, two spherical triangles will be formed corresponding to the two supplementary triedrals. By construction the edge of one triedral is perpendicular to the opposite face of the other, and the vertex of each angle of one triangle is the pole of the opposite side of the other.

Triangles thus formed are called *Polar Triangles,* or Supplementary Triangles. They are otherwise defined thus: If from the vertices of a spherical triangle, as poles, arcs of great circles are described, a spherical triangle is formed, called the *Polar Triangle* of the first.

1. When one triangle is the polar of another, the second is the polar triangle of the first.

2. In two polar triangles each side of the one is the supplement of the opposite angle of the other.

Measurement of Spherical Areas and Volumes.

The methods of obtaining the surface and volume of a sphere have been given, and it now remains to find the surface and volume of parts of a sphere, such as lines, triangles, polygons, zone sections, and segments. The principles pertaining to these several kinds of figures will now be given.

1. The surface or volume of a lune is to the surface or volume of the sphere as the angle of the lune is to 360°.

2. The area of a spherical triangle is equal to the excess of the sum of its angles over two right angles.

3. The volume of a spherical triangle is to the volume of its sphere as the area of its base is to the surface of the sphere.

4. The area of a spherical polygon is measured by the sum of its angles minus the product of two right angles multiplied by the number of sides of the polygon less two; and the vol-

ume of any spherical polygon is to the volume of the sphere as the area of its base is to the surface of the sphere.

5. The area of a zone is equal to the product of its altitude of the circumference of a great circle.

6. The volume of a spherical sector is equal to the area of the zone that forms its base multiplied by one third of the radius of the sphere.

7. The volume of a spherical segment is equal to the half sum of its bases multiplied by its altitude plus the volume of a sphere of which that altitude is the diameter.

8. The volume of a spherical triangle, polygon, pyramid, or sector is equal to the area of its base multiplied by one third of the radius of the sphere.

9. Two spheres are to each other as the cubes of their like dimensions.

Exercises.

1. The entire surface of a sphere is 36 square feet: what is the surface of a lune of this sphere, whose angle is 45°?

2. The angles of a spherical triangle are 60°, 70°, and 80°; if the radius of the sphere be 10 feet, what is the area of the spherical triangle?

3. What would be the area of a spherical polygon on the same sphere as No. 2, provided its angles are 110°, 120°, 150°, and 160°?

4. If the diameter of the earth be 7912 miles, find the surface of the torrid zone if its altitude be 3200 miles, and the earth is regarded as a sphere.

5. In a sphere whose radius is r, find the height of a zone whose area is $2\pi r$.

6. A globe is 10 feet in diameter: how much of its surface could a squirrel see if placed 5 feet above it?

7. The altitude and volume of a spherical segment of one base are given, to find the diameter of the sphere.

8. The surface of a sphere can be completely covered with either 4, 8, or 20 equilateral spherical triangles.

Demonstrating the Formulas.

One of the most valuable exercises is to have the pupils demonstrate all the geometrical formulas, particularly those pertaining to the circle, pyramids, cylinder, cone, sphere, and parts of the sphere. If, at any time, the teacher discovers that a pupil is weakening in his knowledge on any particular point, give him a problem to solve that will cause him to go over the entire field and strengthen himself. Knowledge, unless called into frequent use, slips away very easily; but when it is once thoroughly anchored in the mind it is seldom or never entirely lost.

Modern Geometry.

The more recent developments in geometry are designated *Modern Geometry*, as distinguished from the pure Euclidian Geometry. The progress has been so great, and the discoveries have been so wide and far-reaching, that the extension now constitutes a new science. As much as can be done here is to call attention to this department, and with the hope that all teachers and students of the Euclidian Geometry will give some attention, at least, to this subject. Many of the discoveries are very simple in character and may be very profitably discussed in elementary treatises. The subjects that may be thus introduced are: 1. *Transversals.* 2. *Harmonic Proportion.* 3. *Anharmonic Ratio.* 4. *Pole and Polar to the circle.* 5. *Reciprocal Polars.* 6. *Radical Axes.* 7. *Centers of Similitude.*

Transversals.

Definition. **A Transversal is a straight line intersecting a system of straight lines.**

Definitions to be learned. 1. Segments. 2. Adjacent Segments. 3. Non-adjacent Segments. 4. Complete Quadrilateral.

Propositions.

1. If a transversal cuts the sides of a triangle or the sides produced, the product of three non-adjacent segments of the sides is equal to the product of the other three segments.

2. Conversely, if three points are taken on the sides of a triangle, or the side produced, so that the product of three non-adjacent segments is equal to the product of the other three segments, then the three points are in the same straight line.

This proposition is a simple test for ascertaining whether three points lie in the same straight line.

3. Three straight lines drawn through the vertices of a triangle and any point in its plane divide the sides so that the product of three non-adjacent segments is equal to the product of the other three segments.

4. Conversely, if three points are taken on the sides of a triangle, or on the sides produced, so that the product of three non-adjacent segments is equal to the product of the other three segments, the three straight lines from the opposite vertices of the triangle meet in one point.

Under the 4th proposition the following may be easily demonstrated:

1. That the *medians* of a triangle meet in a point.
2. That the bisectors of the angles of a triangle meet in a point.
3. That the perpendiculars let fall from the vertices of a triangle meet in a point.

Harmonic Proportion.

Definitions.

1. **Harmonic Proportion** is defined by an explanation, thus: Three quantities are in Harmonic Proportion when the first is to the third as the difference between the first

and second is to the difference between the second and third.

2. **A Pencil** is a system of lines radiating from a point in a plane.

3. **A Ray** is one line of a pencil.

4. The point from which the rays start is the **Vertex.**

5. **An Harmonic Pencil** is one which cuts its transversals harmonically.

6. The alternate rays of an harmonic pencil are called **Conjugate.**

7. The four points in which a line is cut by an harmonic pencil are **Harmonic Points.**

8. Two circles are said to intersect *orthogonally* when their tangents at the common point of intersection form a right angle.

Propositions.

1. If a line is divided harmonically, the distance between the points of division is the harmonic mean between the distances from the ends of the line to the point of division not between them.

2. If a straight line AB is divided harmonically at the points C and D, then half of AB is a mean proportional between the distances of its middle point O from the points C and D. Or to prove the relation $OB = OC \times OD$. The converse of this proposition is also true.

3. All transversals of a harmonic pencil are divided harmonically at the points of intersection.

Anharmonic Ratio.

Definitions.

1. **The anharmonic ratio of four points in a straight line,** in which A, B, C, D represent the points in the order of their arrangement, may be expressed thus: $AC \times BD \div AD \times BC$.

2. **The anharmonic ratio of a pencil of four rays is the**

anharmonic ratio of the four points of intersection on these rays as determined by any transversal.

3. **The anharmonic ratio of four fixed points in the circumference** of a circle is the anharmonic ratio of the pencil formed by joining these four points with any point in the circumference.

4. **The anharmonic ratio of four tangents** is the anharmonic ratio of the points of four tangents by a fifth tangent.

Propositions.

1. The anharmonic ratio of four points is not changed by changing two of the letters, provided the other two letters are changed at the same time.

2. If a pencil of four rays be cut by a transversal, any harmonic ratio of the four points of intersection is constant.

3. The anharmonic ratio of four fixed points in the circumference of a circle is *constant.*

4. The anharmonic ratio of four fixed tangents to a circle is constant.

5. When two pencils have the same anharmonic ratio and a common homologous ray, the intersections of the other three pairs of homologous rays are in the same straight line.

6. If two straight lines have the same anharmonic ratio, and a homologous point common, the straight lines joining the other three pairs of homologous points meet in the same point.

7. The intersections of the three pairs of opposite sides of an inscribed hexagon are in the same straight line.

8. The three diagonals which join the opposite vertices of a circumscribed hexagon meet in the same point.

Remark. Proposition 7th is known as Pascal's Theorem, and the 8th as Brianchon's. They are two of the most

beautiful theorems in geometry. From them are derived several other interesting propositions that will well repay the student for the time devoted to them.

Pole and Polar to a Circle.

Definitions.

1. A point is the *pole* of a fixed line, and a line is the *polar* of a fixed point with reference to a circle, if the line and point are so located that the chord of a revolving secant through the point is divided harmonically at the fixed point and the intersection of the fixed line with the secant. Or, if from a fixed point P a line be drawn to C, the center of a circle, and on the line CP a point p be taken such that $CP \times Cp$ is equal to the square of the radius of the circle, then the straight line Dp perpendicular to the line Cp at p is the *polar* of P with respect to the circle, and the point P is the pole of Dp.

2. The intersection of the polar with the diameter through the pole is the *polar point.*

Propositions.

1. Any straight line drawn through the pole to meet a circle is divided harmonically by the circle and the polar.

2. The polar of a given point with respect to a circle is a straight line perpendicular to the diameter drawn through the given point.

Corollaries.

1. The radius of the circle is a mean proportional between the distances of the pole and its polar from the center.

2. If the pole is without, its polar is the chord joining the points of tangency of the tangents drawn to the pole.

3. If the pole is on the circumference, its polar is the tangent at the pole.

4. The pole and polar are interchangeable.

5. The polars of all points of a straight line pass through the pole of that line; and the poles of all straight lines which pass through a fixed point are situated on the polar of that point.

6. The pole of a straight line is the intersection of the polars of any two points.

7. The polar of any point is the straight line joining the poles of any two straight lines through that point.

Reciprocal Polars.

Definitions.

1. Reciprocal polars are two polygons so related that the vertices of either are respectively the poles of the sides of the other with respect to the same circle, which is called the *auxiliary circle.*

2. A theorem inferred from another by means of reciprocal polars is called a *Reciprocal Theorem.*

As examples of reciprocal polars, Brianchon's Theorem may be inferred from Pascal's Theorem, or Pascal's inferred from Brianchon's.

Propositions.

1. The angle contained by two straight lines is equal to the angle contained by the straight lines joining their poles to the center of the auxiliary circle.

2. The ratio of the distances of any two points from the center of the auxiliary circle is equal to the ratio of the distances of each point from the pole of the other.

Radical Axis.

Definitions.

1. **The Power of a Point** in the plane of a circle is the rectangle of the segments, external or internal, into which the point divides the chord passing through it.

5. **The Radical Axis** of two circles is the locus of the point whose powers with respect to the circles are equal.

Propositions.

1. Find the locus of a point from which tangents drawn to two given circles are equal.

This proposition demonstrates that the radical axis of two circles is a straight line perpendicular to the line joining the centers of the circles, and dividing this line so that the difference of the squares of the two segments is equal to the difference of the squares of the radii.

Remark. If the two circles have no point in common, the radical axis does not intersect either of them; if the circles are tangent, externally or internally, their common tangent is their radical axis; and if they intersect, their common chord is their radical axis.

2. The radical axes of a system of three circles, taken two and two, meet in a point.

Centers of Similitude.

Definitions.

The External and Internal Centers of Similitude of two circles are two points which divide the line joining their centers harmonically in the ratio of the two radii.

Propositions.

1. If in two circles two parallel radii are drawn, one in each circle, the straight line joining their extremities intersects the line of centers in the external center of similitude if the parallel radii are in the same direction, and in the internal center of similitude if these radii are in opposite directions.

2. The external centers of similitude of three circles, taken two and two, are in a straight line.

The preceding definitions and propositions will enable the student to obtain some insight into this branch of Geometry. It has been touched but slightly in American text-books, although many valuable contributions have appeared in the various mathematical magazines during recent years.

Here is an excellent field for some American author to occupy.

Conclusion.

The teacher should select the exercises for his classes with great care. Step by step a class can be led to solve problems of great difficulty. By encouraging and not discouraging, the highest proficiency is attained. Never let a pupil lose confidence in himself. The genius of sticking to a problem till light begins to dawn is the chief element in mathematical skill.

Suppose that a teacher wishes to lead his class up to such a problem as this: "To describe a circle that shall touch three given circles." He must start them at first with the simplest exercise, and lead upward gradually, thus:

1. To describe a circle that shall pass through two given points and be tangent to a given straight line.

2. To describe a circle that shall pass through two given points and be tangent to a given circle.

3. To describe a circle that shall pass through a given point and be tangent to two given straight lines.

4. To describe a circle that shall pass through a given point and shall touch a given straight line and a given circle.

5. To describe a circle that shall touch a given circle and be tangent to a given line at a given point.

6. To describe a circle that shall pass through a given point and be tangent to two given circles.

7. To describe a circle that shall be tangent to two given circles and a given straight line.

8. To describe a circle that shall be tangent to three given circles.

If the learner begins with the first problem after he has had the necessary training, and solves it, discussing all possible cases, and so on with each of the others in detail, the 8th will present little difficulty. When he reaches the *Modern Geometry,* he will then discover another and very much simpler method of demonstration than the one outlined in the Euclidian Geometry.

De Quincey says: The act of learning a science is good, not only for the knowledge which results, but for the exercise which attends it: the energies which the learner is obliged to put forth are true intellectual energies; and his very errors are full of instruction. He fails to construct some leading idea, or he even misconstructs it: he places himself in a false position with respect to certain propositions; views them from a false center; makes a false or an imperfect antithesis; apprehends a definition with insufficient rigor, or fails in his use of it to keep it self-consistent. These and a thousand other errors are met by a thousand appropriate resources—all of a true intellectual character: comparing, combining, distinguishing, generalizing, subdividing, acts of abstraction and evolution, of synthesis and analysis, until the most torpid minds are ventilated and healthily excited by this introversion of the faculties upon themselves.